Derivation of the Wheeler-DeWitt equation

Consider the dynamics of the ADM-decomposed metric, which is described by the Einstein-Hilbert action, which in ADM variables gives

$$S[q_{ab}, N_a, N] = \int d^d x\, \mathcal{L} = \int d^d x\, \sqrt{q}\, N\, \left(R[q] + K_{ab}K^{ab} - K^2\right),$$

where the extrinsic curvature is defined as

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - D_{(a}N_{b)}\right), \quad K = q^{ab}K_{ab}.$$

Canonical momenta

We proceed with the canonical analysis of General Relativity. First, we introduce the canonical momenta

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = 2\sqrt{q}N\left(K^{cd} - Kq^{cd}\right)\frac{\partial K_{cd}}{\partial \dot{q}_{ab}} = \sqrt{q}\left(K^{ab} - Kq^{ab}\right).$$

These transform under spatial diffeomorphisms as tensor densities, meaning that the true tensor is

$$p^{ab} = \frac{\pi^{ab}}{\sqrt{q}} = K^{ab} - Kq^{ab}.$$

The next step would be to invert the relation $\pi(q, \dot{q})$ and obtain the generalized velocity $\dot{q}_{ab}$ as a function on the phase space. Actually, since $\dot{q}_{ab}$ is only present in the Lagrangian as a part of K_{ab}, it is even better to obtain K_{ab} as a function on the phase space, and that is what we do next:

$$p^{ab} = K^{ab} - Kq^{ab},$$
$$q \cdot p = q_{ab}p^{ab} = q_{ab}K^{ab} - Kq_{ab}q^{ab} = K - K(d-1) = K(2-d),$$
$$K = \frac{q \cdot p}{2-d},$$
$$K^{ab} = p^{ab} + Kq^{ab} = p^{ab} + \frac{q \cdot p}{2-d}q^{ab}.$$

Phase space Lagrangian

The next step is to cast the Lagrangian into a function of the phase space:

$$L = \int d^{d-1}x\, \mathcal{L} = \int d^{d-1}x\, \sqrt{q}\, N\, \left(R[q] + \left(p_{ab} + \frac{q \cdot p}{2-d}q_{ab}\right)\left(p^{ab} + \frac{q \cdot p}{2-d}q^{ab}\right) - \left(\frac{q \cdot p}{2-d}\right)^2\right) =$$

$$= \int d^{d-1}x\, \sqrt{q}\, N\, \left(R[q] + p \cdot p + 2\frac{(q \cdot p)^2}{2-d} + \left(\frac{q \cdot p}{d-2}\right)^2(d-1) - \left(\frac{q \cdot p}{2-d}\right)^2\right) =$$

$$= \int d^{d-1}x\, \sqrt{q}\, N\, \left(R[q] + p \cdot p - \frac{(q \cdot p)^2}{d-2}\right).$$

Hamiltonian and constraints

Now we calculate the ADM Hamilton's function:

$$H = \int d^{d-1}x\, \mathcal{H} = \int d^{d-1}x\, \left(\pi^{ab}\dot{q}_{ab} - \mathcal{L}\right).$$

First, for the generalized velocities we have

$$\dot{q}_{ab} = 2NK_{ab} + D_{(a}N_{b)} = 2N\left(p_{ab} + \frac{q \cdot p}{2-d}q_{ab}\right) + D_{(a}N_{b)}.$$

The Hamiltonian renders:

$$H = \int d^{d-1}x \left[\sqrt{q} p^{ab} \cdot \left(2N \left(p_{ab} + \frac{q \cdot p}{2-d} q_{ab} \right) + D_{(a} N_{b)} \right) - \sqrt{q} N \left(R[q] + p \cdot p - \frac{(q \cdot p)^2}{d-2} \right) \right] =$$

$$= - \int d^{d-1}x \left[N_a \left(2\sqrt{q} D_b p^{ab} \right) - N\sqrt{q} \left(2p \cdot p + 2\frac{(q \cdot p)^2}{2-d} - R[q] - p \cdot p + \frac{(q \cdot p)^2}{d-2} \right) \right] =$$

$$= - \int d^{d-1}x \left[N_a \left(2\sqrt{q} D_b p^{ab} \right) - N\sqrt{q} \left(p \cdot p + \frac{(q \cdot p)^2}{2-d} - R[q] \right) \right] =$$

$$= - \int d^{d-1}x \left[N_a \cdot C^a(q,p) - N \cdot C(q,p) \right],$$

where

$$C^a(q,p) = 2\sqrt{q} D_b p^{ab} = 2 D_b \pi^{ab},$$

$$C(q,p) = \frac{1}{\sqrt{q}} \left(\pi \cdot \pi - \frac{(q \cdot \pi)^2}{d-2} \right) - \sqrt{q} \cdot R[q] =$$

$$= \frac{1}{\sqrt{q}} \left[\left(q_{ac} q_{bd} - \frac{1}{d-2} \cdot q_{ab} q_{cd} \right) \pi^{ab} \pi^{cd} - \det q \cdot R[q] \right].$$

We observe that the canonical formulation of General Relativity is "frozen" (the Hamiltonian is absent). The evolution of the gravitational field is encoded in d constraints $C^a(q,p) = 0$ and $C(q,p) = 0$.

Diffeomorphism constraints

Consider now the Hamilton-Jacobi formulation of General Relativity. We introduce the Hamilton's functional $S[q_{ab}(x)]$ of the spatial metric, which value is equal to the Einstein-Hilbert action evaluated at the solution of Einstein's equations with boundary metric $q_{ab}(x)$.

The conjugate momenta at the boundary are given by the usual relation

$$\pi^{ab}(x) = \frac{\delta S[q]}{\delta q_{ab}(x)}.$$

The dynamics is encoded in the constraints. First, we examine how the condition $C^a(x) = 0$ influences the possible choices of $S[q]$. For that, lets consider the variation of S under an infinitesimal spatial diffeomorphism:

$$\delta S = \int d^{d-1}x \, \frac{\delta S[q]}{\delta q_{ab}(x)} \, \delta q_{ab}(x) = \int d^{d-1}x \, \pi^{ab}(x) \, \left(D_a \varepsilon_b + D_b \varepsilon_a \right) =$$

$$= - \int d^{d-1}x \, \varepsilon_a \left(2 D_b \pi^{ab} \right) = - \int d^{d-1}x \, \varepsilon_a(x) \cdot C^a(x).$$

Hence the vanishing of C^a is equivalent to postulating the diffeomorphism-invariance of $S[q]$. The meaning of the constraints associated with the shift vector N_a from the ADM formalism is straightforward: the Hamilton's functional for General Relativity has to be defined on distinct Riemannian geometries instead of metric fields.

Wheeler-DeWitt

The remaining constraint is the one associated with the lapse function: $C = 0$. Its physical meaning is the description of evolution of the gravitational field in time, although there is no external notion of time. Time coordinate does not correspond to physical time in General Relativity. Instead, the physical time is determined by the gravitational field itself.

The Hamilton-Jacobi equation for the Hamilton's functional $S[q]$ is given by the vanishing of the Hamiltonian constraint $C = 0$:

$$\left(q_{ac} q_{bd} - \frac{1}{d-2} \cdot q_{ab} q_{cd} \right) \frac{\delta S}{\delta q_{ab}} \cdot \frac{\delta S}{\delta q_{cd}} - \det q \cdot R[q] = 0.$$

As a first step towards the quantization of General Relativity, it is possible to consider this equation as an eikonal approximation of the wave equation for the wavefunction of the Universe $\Psi[q]$:

$$\left[\hbar^2 \cdot \left(q_{ac}q_{bd} - \frac{1}{d-2} q_{ab}q_{cd}\right) \frac{\delta^2}{\delta q_{ab}\, \delta q_{cd}} + \det q \cdot R[q]\right] \Psi[q] = 0,$$

or, in natural units for 4 spacetime dimensions,

$$\left[\left(q_{ac}q_{bd} - \frac{1}{2} q_{ab}q_{cd}\right) \frac{\delta^2}{\delta q_{ab}\, \delta q_{cd}} + \det q \cdot R[q]\right] \Psi[q] = 0 \qquad (d = 4, \hbar = 1).$$

This is known as Wheeler-DeWitt equation. It relates to the Hamilton-Jacobi-Einstein equation in the same way as Schrodinger's equation relates to the Hamilton-Jacobi equation for the Newtonian particle: by introducing the quantum operator for canonical momenta in the coordinate basis,

$$\hat{\pi}^{ab}(x)\Psi[q] = -i\hbar \frac{\delta \Psi[q]}{\delta q_{ab}(x)}.$$

As everything which has to do with quantum gravity, the WdW equation is not well-defined mathematically, since it contains second functional derivatives. It is also difficult to deal with since it is defined on the superspace (a space of all spatial geometries).

These problems can be resolved in the framework of LQG. But even without loops, quantum General Relativity can be used in the symmetry-reduced cosmological models with finite-dimensional "minisuperspace".