Ward identity in QM

Consider a free nonrelativistic 1-dimensional quantum particle:

$$S[x] = \int dt \frac{m\dot{x}^2}{2}.$$

Suppose we are interested in a quantum evolution equation, thus we are considering time translations:

$$\delta S = 0 = \int dt \frac{dL}{dt} \cdot \varepsilon = \int dt \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right)\varepsilon.$$

We can pass from global to local transformations:

$$\delta S = \int dt \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right)\varepsilon(t).$$

Now the action is not invariant under a local time translation. The corresponding Noether charge is the energy of the particle

$$E = \frac{m\dot{x}^2}{2}.$$

We now derive the Ward identity from the invariance of the path integral:

$$\langle \delta F \rangle = i \langle F \delta S \rangle .$$

Take

$$F[x] = x(t_0) \cdot F(t_1, t_2, \dots)$$

for some F which depends on times t_i each lying outside some vicinity of t_0. Consider a special case of $\epsilon(t)$:

$$\varepsilon(t) = \begin{cases} 1 & |t - t_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\delta F = \delta x(t_0) \cdot F(t_1, t_2, \dots)$$

$$F\delta S = x(t_0) \cdot F(t_1, t_2, \dots) \cdot \int_{t_0-\epsilon}^{t_0+\epsilon} dt \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right) = x(t_0) \cdot F(t_1, t_2, \dots) \cdot \left.\frac{m\dot{x}^2}{2}\right|_{t_0-\epsilon}^{t_0+\epsilon}.$$

We thus have a Ward identity:

$$\forall F : \quad \langle \delta x(t_0) \cdot F(t_1, t_2, \dots) \rangle = i \left\langle x(t_0) \cdot F(t_1, t_2, \dots) \cdot \left.\frac{m\dot{x}^2}{2}\right|_{t_0-\epsilon}^{t_0+\epsilon} \right\rangle .$$

Since this is true for any Heisenberg operator F, we can say that the operator equation is obeyed:

$$\delta \hat{x}(t_0) = i\hat{x}(t_0) \cdot \left.\frac{m\dot{\hat{x}}^2}{2}\right|_{t_0-\epsilon}^{t_0+\epsilon} = i\left(\hat{H}\hat{x} - \hat{x}\hat{H}\right),$$

where the automatic time-ordering in the path integral gives the commutator of Heisenberg operators. Hence we get the Heisenberg equation for time evolution:

$$\delta_t \hat{x} = i\left[\hat{H}, \hat{x}\right].$$