Vacuum energy normal-ordering

The present note proves that the canonical Hamiltonian can be automatically rendered normal-ordered by considering an infinite cosmological constant term in the bare action and tuning it in such a way that the partition function in the absense of sources is equal to one: $Z[0] = 1$.

Consider the simplest possible example of a free field theory — the $1+0$ dimensional free scalar theory spawned by the action

$$S[X] = \int dt \left\{ \frac{1}{2}\dot{X}^2 - \frac{m^2}{2}X^2 - \Lambda \right\} = \frac{1}{2} \langle X | \hat{\Theta} | X \rangle - W,$$

where

$$|Y\rangle = \delta\left(Y - X\right), \quad \langle X|Y\rangle = \int dt\, X^*(t)\, Y(t),$$

$$\hat{\Theta} = -\left(\frac{d^2}{dt^2} + m^2\right) = \int \frac{d\omega}{2\pi} \left(\omega^2 - m^2 + i0\right) |\omega\rangle \langle\omega| ,$$

$$W = \Lambda \cdot \int dt = \Lambda \cdot \delta(0),$$

and by $\delta(0)$ we denote the time (infinite) volume of 1-dimensional time. The functional integral for the partition function in the presence of sources is

$$Z[J] = \int DX \exp\left\{ \frac{i}{2} \langle X | \hat{\Theta} | X \rangle - iW + i \langle J|X\rangle \right\} =$$

$$= e^{-iW} \cdot \left(\det \frac{\hat{\Theta}}{2\pi i} \right)^{-1/2} \cdot \exp\left\{ -\frac{i}{2} \langle J | \hat{\Theta}^{-1} | J \rangle \right\} =$$

$$= \exp\left\{ -iW - \frac{1}{2}\mathrm{tr}\log\frac{\hat{\Theta}}{2\pi i} - \frac{i}{2} \langle J | \hat{\Theta}^{-1} | J \rangle \right\}.$$

We calculate the functional determinant in the r.h.s. of this expression by means of zeta-function regularization:

$$\mathrm{tr}\log\frac{\hat{\Theta}}{2\pi i} = \delta(0) \cdot \int \frac{d\omega}{2\pi} \log\left[\frac{\omega^2 - m^2 + i0}{2\pi i} \right] = -\delta(0)\cdot z'(0),$$

$$z(s) = \int d\omega \left(\frac{2\pi i}{\omega^2 - m^2 + i0} \right)^s = -i \int d(i\omega) \left(\frac{2\pi i}{-(i\omega)^2 - m^2} \right)^s =$$

$$= \frac{i\sqrt{\pi} m^{1-2s} (-1)^{s+1} \Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)},$$

$$\mathrm{tr}\log\frac{\hat{\Theta}}{2\pi i} = -\delta(0)\cdot z'(0) = im \cdot \delta(0),$$

$$Z[0] = \exp\left\{ -iW - \frac{1}{2} im \cdot \delta(0) \right\} = \exp\left\{ -i\delta(0) \left(\Lambda + \frac{m}{2} \right) \right\}.$$

The renormalization condition

$$Z[0] = e^{-i\delta(0)\cdot\Lambda_R}$$

stands when the bare coupling is taken to be

$$\Lambda = \Lambda_R - \frac{m}{2}.$$

In the second part of this note we compute the vacuum expectation of the field energy at time t:

$$E(t) = \frac{1}{2}\dot{X}(t)^2 + \frac{m^2}{2}X(t)^2 + \Lambda,$$

$$\langle E \rangle = \left(-\frac{1}{2}\frac{d^2}{ds^2} + \frac{m^2}{2} \right) F(s)\Big|_{s=0} + \Lambda,$$

where the propagator

$$F(s) = \langle X(t) \cdot X(t+s) \rangle = \int \frac{d\omega}{2\pi} \frac{i \cdot e^{i\omega s}}{\omega^2 - m^2 + i0} =$$

$$= -i \int \frac{d(i\omega)}{2\pi} \frac{i \cdot e^{(i\omega)s}}{-(i\omega)^2 - m^2} = -\frac{e^{-im\cdot|s|}}{2m}.$$

$$F(s)|_{s=0} = -\frac{1}{2m}, \quad \frac{d^2}{ds^2} F(s)\Big|_{s=0} = \frac{m}{2},$$

$$\langle E \rangle = \frac{m}{4} + \frac{m}{4} + \Lambda = \frac{m}{2} + \Lambda = \frac{m}{2} + \Lambda_R - \frac{m}{2} = \Lambda_R.$$

Note that a particular case of the renormalization condition $Z[0] = 1$ corresponds to the zero vacuum energy $\langle E \rangle = 0$ and thus renders the canonical Hamiltonian to be automatically normal-ordered.

Since the same logic applies to quantum fields in higher spacetime dimensions (as they can be described by collections of harmonic oscillators), the normal-ordering of the canonical Hamiltonian of the field theory corresponds to an infinite cosmological term which has to be tuned exactly such that the renormalization condition $Z[0] = 1$ holds.