TOCY spinfoams

Spinfoam models are covariant (Lagrangian) fully background-independent quantizations of diffeomorphism-invariant classical field theories inspired by Feynman path integrals. These include a range of models, some of which have ignited more or less physical interest. Spinfoams are related to GFTs — QFTs on the Lie group as the base manifold by a perturbative duality. This duality is conjectured to define nonperturbatively transition amplitudes for spinfoam models.

Spinfoams are a tool for calculating transition amplitudes for spin networks, which form an orthogonal basis in the Hilbert space of the background-independent quantum BF theory. For an introduction on spin networks in the context of General Relativity and Loop Quantum Gravity, see this post.

In the present post I present the TOCY (Turaev-Ooguri-Crane-Yetter) spinfoam model, which can be considered a quantization of a certain topological field theory called BF theory. In 3D spacetime BF theory reduces to General Relativity and the quantization of 3D relativity is given by a special case of TOCY spinfoams: Ponzano-Regge spinfoams.

In 4D spacetime BF theory is in many aspects similar to General Relativity. An appropriate adoptation of the TOCY model to 4D General Relativity is called the Barrett-Crane spinfoam model. Essentialy it is the covariant approach to calculating transition amplitudes of Loop Quantum Gravity. I hope to cover it in another post anytime soon.

Classical BF theory

Our starting point is a diffeomorphism-invariant classical field theory. Diffeomorphism invariance plays a crucial role here: the theory can't be defined with respect to background spacetime (like Maxwell's theory, for example). Instead the classical solutions of the equations of motion have to correspond to different classical spacetimes. This is exactly the situation with General Relativity.

In this section we will consider another theory: the d-dimensional BF theory. It is given by the action over the base manifold M:

$$S_{BF,d}[A,B] = \mathrm{tr} \int_M B \wedge F, \qquad F = dA + A \wedge A,$$

thus the name. The fields A and B entering the action have the following geometrical interpretation:

- Gauge connection A is an 1-form taking values in the Lie algebra $\mathfrak{g}$ of the gauge group G,
- Lagrange multiplier B is an $(d-2)$-form which also takes values in $\mathfrak{g}$,
- tr is the invariant trace (Killing form) on $\mathfrak{g}$: $\mathrm{tr}(ab) = \mathrm{tr}(\mathrm{ad}(a) \cdot \mathrm{ad}(b))$.

There are two examples which are particularly worth mentioning:

- The $d = 3$ BF theory is defined on the 3-dimensional manifold M. In this case B is an 1-form and the action reads

  $$S_{BF,\, d=3} = \frac{1}{2} \int_M \underline{B}^a \wedge \underline{\underline{F}}^a.$$

  By choosing the gauge group G to be $SO(3)$ or $SO(2,1)$ we get the Euclidean and Lorentzian versions of 3d General Relativity in Palatini variables respectively (the 1-form B becomes the frame field e and the gauge connection becomes the spin connection ω).

- The $d = 4$ BF theory is defined on the 4-dimensional manifold M. in this case B is a 2-form and the action reads

  $$S_{BF,\, d=4} = \frac{1}{2} \int_M \underline{\underline{B}}^a \wedge \underline{\underline{F}}^a.$$

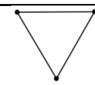
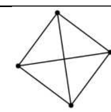
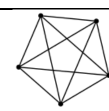
  By choosing the gauge group G to be $SO(4)$ or $SO(3,1)$ we get something close to the Euclidean and Lorentzian versions of 4d General Relativity in Palatini variables respectively. Remember the Palatini action for General Relativity?

  $$S_{\mathrm{Palatini}}[e,\omega] = \frac{1}{2} \int_M \underline{e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}_{IJ}, \qquad \underline{\underline{R}} = \underline{\underline{d\omega}} + \underline{\omega} \wedge \underline{\omega}.$$

The only difference is that we have a 2-form $\underline{\underline{B}}^{IJ}$ in place of the expression $\underline{e}^I \wedge \underline{e}^J$. As it turns out, there is a major difference between these two theories. For instance, BF theory is topological (has no local degrees of freedom) in any number of dimensions, whereas General Relativity admits two graviton polarizations in 4 spacetime dimensions. Nevertheless, later we will be able to adopt the spinfoam quantization of BF theory to General Relativity (Barrett-Crane spinfoam model) by imposing an additional constraint on configuration variables: the so-called *simplicity constraint*, which ensures that $\underline{\underline{B}}^{IJ}$ is not arbitrary, but can be expressed as $\underline{e}^I \wedge \underline{e}^J$ for some frame field $\underline{e}^I$.

Spacetime triangulation

Next step is to consider a triangulation of the base differential manifold M into a simplicial manifold Δ. The elements of the triangulation are called simplexes. A simplex of order n is a full unoriented graph with $n+1$ vertices. Simplices of low order have special names:

Rank	Simplex name	Simplex graph
$n = 0$	*point*	
$n = 1$	*segment*	
$n = 2$	*triangle*	
$n = 3$	*tetrahedron*	
$n = 4$	*pentachoron*	

Each rank-n simplex is bounded by exactly $n+1$ rank-$(n-1)$ simplexes. E.g. a segment is bounded by two points, a triangle is bounded by three segments, etc. Any bounding subsimplex contains all of the vertices of the original simplex except for one. The excluded vertex completely determines the bounding subsimplex.

In order to triangulate an d-dimensional Riemannian (or Lorentzian) manifold we need a family of simplexes of ranks all the way from 0 to d. E.g. a surface is triangulated into triangles, some of which share bounding segments, some of which share bounding points. Consequently, we only need five simplexes drawn above to triangulate a 4-dimensional spacetime of General Relativity.

For any triangulation Δ of an d-dimensional differential manifold M there exists a dual skeleton Δ^*. The dual rank n^* of the dual simplex and the rank of the simplex satisfy

$$n + n^* = d.$$

Elements of Δ^* are also given shorthand names for convenience:

Dual rank	Dual simplex name	Dual to (in 3D)	Dual to (in 4D)
$n^* = 0$	*vertex*	tetrahedron	pentachoron
$n^* = 1$	*edge*	triangle	tetrahedron
$n^* = 2$	*face*	segment	triangle
$n^* = 3$	*region*	point	segment

Note that Δ^* is not a triangulation! For example, the rank-2 dual simplex, also known as a face, is not in general a triangle, but an arbitrary polygon with k nodes, k unbounded.

However, there are some constraints on Δ^* coming from the fact that Δ is a triangulation. For example, since in 3D tetrahedrons are bounded by 4 triangles and in 4D pentachorons are bounded by 5 tetrahedrons, the number of dual edges, incident to the same dual vertex (also called the valency of the vertex) is equal to 4 in 3D and 5 and 4D (in general, it is equal to $d+1$).

Also, the number of faces incident to an edge is equal to d. Indeed, in 3D edges and faces are dual to triangles and segments, and there are 3 segments bounding a triangle. In 4D they are dual to tetrahedrons and triangles and there are 4 triangles bounding a tetrahedron.

Discretization of variables

The next step is to discretize the variables of the classical BF theory on the triangulation. We associate to the following discrete variables to the edges and faces of the dual skeleton Δ^*:

1. To each dual edge e we associate a holonomy of the gauge connection along the edge:

$$G \ni g_e = \mathrm{P}\exp \int_e \underline{A}.$$

2. To each dual face f we associate the $(d-2)$-form B, integrated along the $(d-2)$-simplex $\sigma_{(d-2)}$ to which the face f is dual:

$$\mathfrak{g} \ni l_f = \int_\sigma B.$$

This choice of discretized variables is pretty natural, but it is not unique, which is expected. Since quantization is highly ambiguous, we are to make untrivial choices. It is passing to the classical limit which has to be unequivocal.

The discretized action for the BF theory on the dual skeleton Δ^* reads

$$S[g,l] = \sum_f \mathrm{tr}\left[g_{e_1} \dots g_{e_k} l_f\right],$$

where $e_1, \dots, e_k$ are the k edges bounding the dual face f (remember that since Δ^* is not a triangulation, k is not necessarily equal to 3). In the context of this post it is to be taken for granted that $S[g,l]$ correctly approximates $S_{BF,d}[A,B]$.

Spinfoam quantization of BF theory

Spinfoam quantization is carried out in the spirit of path integral quantization. Consider a formal expression for the path integral:

$$Z = \int Dg \int Dl \cdot e^{iS[g,l]}.$$

The choice of the path integral measure is dictated by gauge invariance: we have to choose

$$Z = \prod_e \int dg_e \cdot \prod_f \int dl_f \cdot e^{iS[g,l]},$$

where dg_e is the Haar invariant measure on the Lie group G and dl_f is given by the Killing form on the algebra $\mathfrak{g}$.

Thus, Z is a well-defined discrete version of the partition function for the BF theory.

Lets do the above integrals. We start by integrating out the Lagrange multipliers l_f:

$$Z = \prod_e \int dg_e \cdot \prod_f \delta(g_{e_1} \dots g_{e_k}),$$

where we have defined the Dirac delta function on the group G as a generalized function through the relation

$$\forall f(g): \quad \int dg\, f(g)\, \delta(g) = f(e)$$

with e the group identity.

Then, we plug in the Peter-Weyl expansion for the delta function:

$$\delta(g) = \sum_\rho \dim\rho \cdot \mathrm{tr} W_\rho(g),$$

where W_ρ are the Wigner functions covered in my previous post. This gives

$$Z = \prod_e \int dg_e \cdot \sum_{\{\rho_f\}} \left(\prod_f \dim\rho_f \cdot \left[W_\rho\right]^\alpha_\alpha (g_{e_1} \dots g_{e_k}) \right) =$$

$$= \prod_e \int dg_e \cdot \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f \cdot [W_\rho]^\beta_\alpha (g_{e_1}) \cdot [W_\rho]^\gamma_\beta (g_{e_2}) \dots [W_\rho]^\alpha_\zeta (g_{e_k})) \right).$$

It is possible to regroup the Wigner functions in terms of edges e. Since Δ is a triangulation, each edge is present in boundaries of exactly d faces. Edge integrals can be evaluated:

$$\int dg_e \, [W_{\rho_1}]^{\beta_1}_{\alpha_1} (g_e) \dots [W_{\rho_d}]^{\beta_d}_{\alpha_d} (g_e) = \sum_i v^i_{\alpha_1 \dots \alpha_d} v^{\beta_1 \dots \beta_d}_i,$$

where i labels the orthogonal basis in the space of intertwiners of $\rho_1 \otimes \dots \otimes \rho_d$.

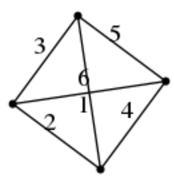
The rest of Z can be computed by noticing that to each dual vertex corresponds a pattern of contractions of intertwiner indices (we have two intertwiners for each dual edge). We will consider two special cases: 3D and 4D.

3D

In 3D, the pattern of contractions is dictated by the tetrahedron to which the vertex is dual:

$$A(\rho_1, \dots, \rho_6, a, b, c, d) = v^{\alpha_3 \alpha_2 \alpha_1}_a v^{\alpha_4 \alpha_6 \alpha_2}_b v^{\alpha_4 \alpha_1 \alpha_5}_c v^{\alpha_3 \alpha_6 \alpha_5}_d,$$

where a, b, c and d label the intertwiners at dual edges (or at triangles), the contractions are done with help of the Killing form on $\mathfrak{g}$ and the pattern of contractions is dictated by the geometry of tetrahedron (to which the vertex is dual):



If $G = SU(2)$, there is always a single choice of intertwiner (a fact from representation theory of $SU(2)$) and the vertex amplitude only depends only on 6 spins $j_1 \dots j_6$. The partition function of this model (Ponzano-Regge spinfoam model) is equal to

$$Z = \sum_{\{j_f\}} \left(\prod_f (2j_f + 1) \cdot \prod_v \{6j\} \right),$$

where the vertex amplitude contraction $A(j_1, \dots, j_6)$ is called the 6j symbol and denoted $\{6j\}$. In general,

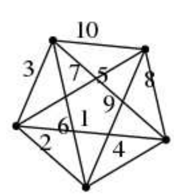
$$Z = \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f \cdot \prod_v \{10j\} \right),$$

where the $\{10j\}$ symbol stands for $A(\rho_1, \dots, \rho_6, a, b, c, d)$ which depends on 6 irreps and 4 intertwiners (total of 10 parameters).

4D

Analogously, in 4D the pattern of contractions is dictated by the pentachoron:

$$A(\rho_1, \dots, \rho_{10}, a, b, c, d, e) = v^{\alpha_3 \alpha_6 \alpha_5 \alpha_2}_a v^{\alpha_2 \alpha_6 \alpha_9 \alpha_4}_b \cdot$$
$$\cdot v^{\alpha_4 \alpha_1 \alpha_7 \alpha_8}_c v^{\alpha_8 \alpha_9 \alpha_5 \alpha_{10}}_d v^{\alpha_{10} \alpha_9 \alpha_6 \alpha_3}_e.$$



The transition amplitude in this case is equal to

$$Z = \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f \cdot \prod_v \{15j\} \right),$$

where the $\{15j\}$ symbol stands for $A(\rho_1, \dots, \rho_6, a, b, c, d, e)$ which depends on 10 irreps and 5 intertwiners (total of 15 parameters).

Transition amplitudes of the quantum BF theory are defined as sums over spinfoams:

Spinfoam is a simplicial 2-complex with

- Faces colored with irreducible representations of the gauge group G,
- Edges colored with elements of the orthonormal basis of intertwiners between the representations of incident faces.

By analogy with spin networks, we can either define spinfoams as colorings of a particular contably-infinite 2-complex, or we can consider faces colored with trivial representations non-existent.
A boundary of the spinfoam is a graph consisting of nodes and links. Nodes come from intersections of edges with the boundary, and are thus colored from intertwiners. Links come from intersections of faces with the boundary and are colored with irreducible representations. There are precisely d-valent spin networks. Thus, *the boundary of a spinfoam is a spin network.*

We define the transition amplitude of the spin network $Z(s)$ to be a sum over all spinfoams which are bounded by s:

$$Z(s) = \sum_{\sigma\colon \partial\sigma = s} \left(\prod_f \dim \rho_f \cdot \prod_v A_v \right)$$

with the amplitude A_v given by a contraction of intertwiners with the pattern of contraction given by the topology of the rank-d simplex.

Bubble divergences

The sum as is stands is ill-defined because of divergences. These arise (by analogy with closed loops in Feynman diagrams) in closed 2-subcomplexes called *bubbles*.

Consider, for example, a simple bubble in the Ponzano-Regge model (3D, $G = SU(2)$). It consists of 4 vertices around a point in the triangulation.

The amplitude can be calculated directly from the spinfoam formalism, provided that external faces are non-existent (colored with $j = 0$):

$$A_{\text{bubble}} = \sum_j (2j+1)^4 \cdot \{6j\}\,(j,j,j,0,0,0)^4 =$$

$$= \sum_j (2j+1)^4 \cdot \left(v^{\alpha\beta} v^{\beta\gamma} v^{\gamma\alpha}\right)^4,$$

where the normalized intertwiner is given by

$$v^{\alpha\beta} = \frac{\delta^{\alpha\beta}}{\sqrt{2j+1}}.$$

This gives

$$A_{\text{bubble}} = \sum_j (2j+1)^4 \cdot \left(\frac{2j+1}{(2j+1)^{3/2}} \right)^4 =$$

$$= \sum_j (2j+1)^2 = \delta_{SU(2)}(0),$$

since l.h.s. is exactly the Peter-Weyl expansion for the $SU(2)$ delta function.

Alternatively, we could use the pre-spinfoam expression:

$$A_{\text{bubble}} = \int dg_1 \dots dg_6 \delta(g_1 g_2 g_6^{-1}) \delta(g_3 g_4 g_6).$$

$$\cdot\delta(g_4 g_1 g_5^{-1})\delta(g_2 g_3 g_5) = \delta_{SU(2)}(0).$$

Here we have used the left-invariance and the normalization condition for the Haar measure:

$$\int dg = 1.$$

This divergence is *infrared* as it occurs when j are large. It contributes to the unphysical normalization factor in the path integral measure and can be divided away by requiring

$$Z(\emptyset) = 1$$

for the empty spin network.

Several regularizations could be applied. For starters, one can always cut off large spins ($j \geq J$), divide by $Z(\emptyset)$ and then pass to the $J \to \infty$ limit.

But there is another kind of regularization which has the advantage of being gauge-invariant: one can pass to the q-deformed quantum group G_q ($SU(2)_q$ in case of Ponzano-Regge). It always has a finite number of irreducibles and in the $q \to 0$ the initial theory is restored.

An interesting fact is that spinfoams exist for small but finite values of q. This is called Turaev-Viro spinfoam model. It corresponds to the BF theory with a cosmological constant term of order q^{-1}.