SU(2) recoupling and triangulation independence

We work in the framework of TOCY spinfoams with $SU(2)$ gauge group. The classical limit of this model is known to be $SU(2)$ BF topological field theory with action

$$S[A,B] = \mathrm{tr}\int_M B \wedge F(a).$$

Binor calculus

Consider the fundamental (spin-1/2) irrep of $\mathfrak{su}(2)$. We will denote objects belonging to this irrep with capital latin indices taking values in $\{0,1\}$, i.e.

$$\psi^A = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}.$$

There are two tensorial objects that play a special role in the $SU(2)$ recoupling theory. The first is the Kronecker delta δ_A^B, which is an always an invariant tensor under any symmetry group. The second is the antisymmetric spinorial Levi-Civita tensor, or simply epsilon-tensor defined with $\varepsilon_{01} = -\varepsilon^{01} = 1$:

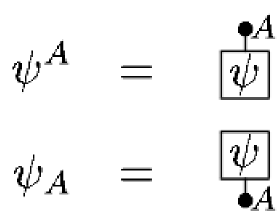
$$\varepsilon_{AB} = -\varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

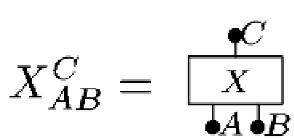
The epsilon-tensor is $SU(2)$-invariant:

$$\forall U \in SU(2): \quad \varepsilon^{AB} U^C_{\ A} U^D_{\ B} = \varepsilon^{CD}.$$

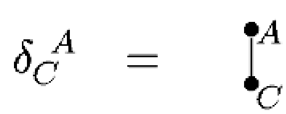
We will use the following incredibly convenient graphical notation, first introduced by Roger Penrose in his doctoral thesis, called *binor calculus*.

Fundamental-irrep indicex contractions are denoted by *wires* (solid curves). Free indices (the ones we aren't summing over) are denoted by a thick point sitting on the end of the wire. Tensors are denoted by *cables* with wires connected to them. Covariant idices correspond to the wires coming out from the bottom of the cable, while contravariant indices correspond to the wires coming out from the top. For example, these are graphical representations of ψ^A, ψ_A, X^C_{AB}:





The Kronecker delta is represented by a straight wire, which means that the contraction of indices is represented by joining the loose ends of wires:



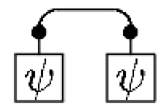
The epsilon-tensors are represented by U-shaped contractions:

$$\epsilon_{AC} =$$ 

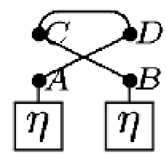
$$\epsilon^{AC} =$$ 

The indices of the epsilon-tensor always come from the left end of the wire to the right.

For example, consider the following contraction:

$$\epsilon_{AB}\psi^A\psi^B =$$ 

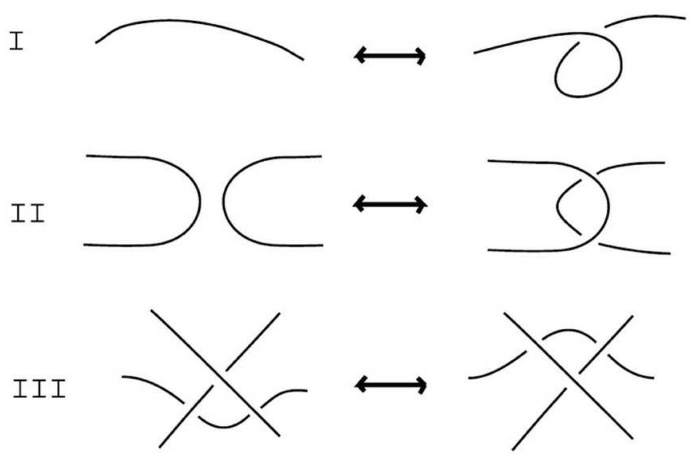
This graphical calculus has one drawback though: it isn't topological. Consider, for example, the same diagram with twisted wire:

$$-\epsilon_{CD}\delta^D_A\delta^C_B\eta^A\eta^B = -$$ 

The diagram on the right can be smoothly transformed into the previous one by a single Reidemeister move of type I (twist). But the expression changes sign!

But it turns out (first pointed out by Penrose) that we can make the graphical calculus topologically-invariant by altering a single rule: we associate a minus sign to each planar intersection of wires. The resulting calculus indeed turns out to be topologically-invariant, meaning that we can smoothly deform the wires, changing the expression that they represent, but the value of this expression will always be equal to the value of the original expression.

In order to prove this, we have to consider all three Reidemeister moves (type I "*twist*", type II "*poke*" and type III "*slide*") which are shown on the picture below:



Exercise: prove this claim by showing it true for each Reidemeister move consequently.

Skein relations and knot polynomials

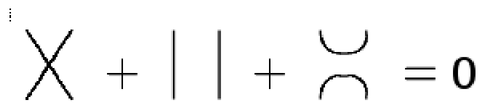
Topologically-invariant calculus on wires is strongly connected to knot polynomials. It is of no surprise: consider configurations with no tensorial objects besides δ^B_A, ε_{AB} or ε^{AB} which doesn't have loose wires. In general, the wires close to give a *link*, or a topological collection of *knots*. The rules of the calculus associate a numeric value for the link.

For example, consider the simplest possible link, which is the 0_1 knot or the *unknot*:

This gives -2 because of our definition of epsilon-tensors:

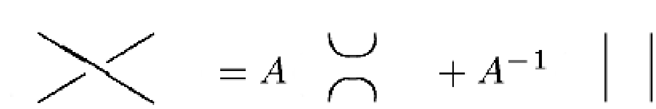
$$\varepsilon_{AB}\varepsilon^{AB} = \varepsilon_{01}\varepsilon^{01} + \varepsilon_{10}\varepsilon^{10} = (-1) + (-1) = -2.$$

Knot polynomials are spawned by a particular algebraic relation called the *skein relation*. In our case, it is easy to prove that



This two facts (the value of the unknot and the skein relation) are enough to calculate the value of any given link. For example, try calculating the value for the trefoil knot, which will turn out to be -2 (note how different knots might accidentally have the same value).

There is a certain modification of the skein relation, the Kauffman relation, which has to do with the quantum group $SU_A(2)$:



Calculating the value of the trefoil knot gives the following knot polinomial:

$$K_{\text{trefoil}}(A) = -2\left(1 + A + \frac{1}{A^2}\right),$$

which for the case $A = -1$ (binor calculus) gives the obtained above -2. Note, however, that the knot polynomial for the trefoil knot differs from the one of the unknot

$$K_{\text{unknot}}(A) = -2$$

for the values of $A \neq -1$.

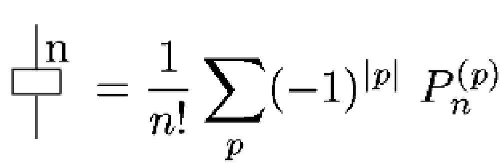
Recoupling theory

We now present a slightly modified form of the binor calculus, which is called the $SU(2)$ *recoupling theory.*

First, we will denote by a straight *multiwire* labeled with n a system of n parallel binor wires. Multiwires correspond to tensor products of n fundamental irreps, and also to spin-j irreps with $j = n/2$ as follows:

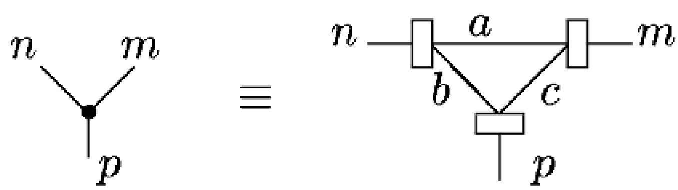
> The spin-j irreducible representation of $\mathfrak{su}(2)$ is isomorfic to the symmetrized tensor product of $n = 2j$ fundamental (spin-1/2) irreps.

An empty cable on the multiwire corresponds to the antisymmetrization in all n indices:



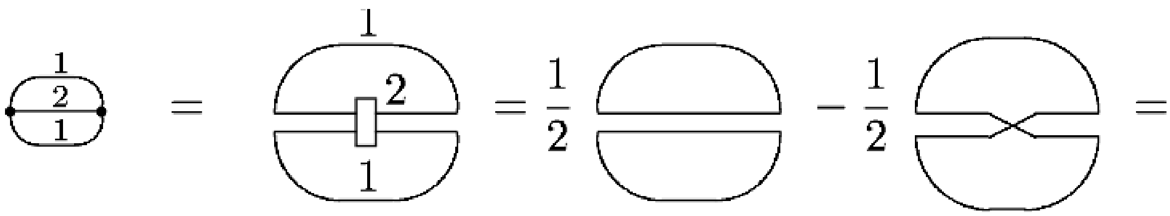
where the sum is taken over the $n!$ of binor wire permutations, $|p|$ is the sign of the permutation p and $P_n^{(p)}$ corresponds to the tangle of permuted binor wires denoted by p.

Intertwiners are denoted by points, connected with multiwires corresponding to the irreps of the intertwiner. The expression for the trivalent intertwiner is given by the recoupling theory graphical notation:



Here we've split the $n = a + b$ binor wires, $m = a + c$ and $p = b + c$. In case there is no such splitting, the Clebsh-Gordon conditions (quantum triangle inequality) are unsatisfied, and there is no intertwiner.

As an example of recoupling theory computation, consider the norm-squared of the 3-valent intertwiner:



$$= \frac{1}{2}(-2)^2 - \frac{1}{2}(-2) = 3.$$

Here we've omitted the (trivial) antisymmetrizer of the single binor wire.

Triangulation independence of 2d BF theory

Consider the spinfoam amplitude of the particularly simple model: 2d $SU(2)$ BF theory given by the action

$$S[\underline{A}, B] = \mathrm{tr} \int_M B \cdot \underline{\underline{F}}(A).$$

The TOCY spinfoam model gives the folowing expression for the quantum amplitude of the spin network s:

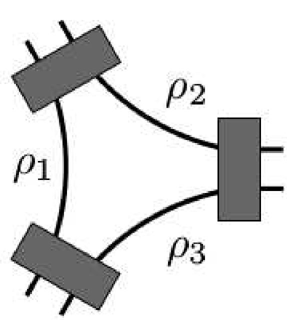
$$Z(s) = \sum_{\{j_f\}} \prod_e \int dg_e \cdot \prod_f \dim j_f \cdot \mathrm{tr}\left[\rho_f \left(g_{e_1} \ldots g_{e_k}\right)\right].$$

In 2 spacetime dimensions, each edge of the spinfoam is bounded by two faces. Thus, the edge integral gives the invariant tensor

$$P^{\alpha_1\alpha_2}_{\beta_1\beta_2}(j_1, j_2) = \int dg_e \left[W_{j_1}\right]^{\alpha_1}_{\ \beta_1}(g_e) \cdot \left[W_{j_2}\right]^{\alpha_2}_{\ \beta_2}(g_e),$$

and the pattern of contractions is given by the topology of the triangle.

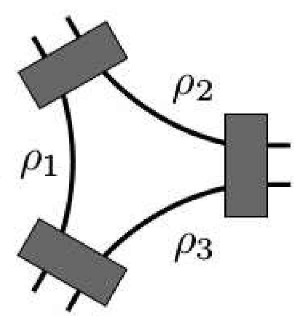
More concretely, by representing $P^{\alpha_1\alpha_2}_{\beta_1\beta_2}(j_1, j_2)$ in the recoupling theory notation by a cable of recoupled wires, we arrive at the following contraction pattern in the spinfoam vertex:



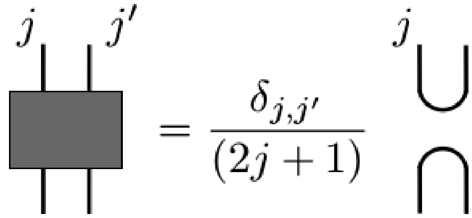
The wires are labeled with irreps and correspond to the three incident faces to the vertex. The projection symbols correspond to the three incident edges.

The partition function of the TOCY model, for a given triangulation Δ of spacetime, is given by

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f:\{f\}\to\rho_f} \; \prod_{f\in\Delta^\star} \mathrm{d}_\rho$$

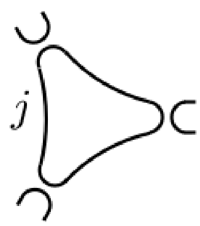


Let's calculate this for a closed surface without boundary. For this we have to give the projection operator in terms of recoupling theory graphical calculations:



The delta symbol $\delta_{j,j'}$ in this expression is necessary because there could be no 2-valent intertwiners unless $j = j'$. The only invariant tensor with two indices is proportional to the epsilon-tensor, and the proportionality coefficient is determined by taking the trace and plugging it into the normalization condition for the intertwiner. This gives a square root $\sqrt{2j+1}$, which gets squared because we've got two U-shaped intertwiners in the projection operator.

Now we can trace how recoupled wires behave in the general spinfoam amplitude. The first observation is: all spins must be equal to each other (otherwise the delta symbol would cancel the amplitude). The second observation is: the generic pattern of the recoupled wires on the 2d surface is given by the following picture:



We arrive at the following conclusion:

1. The contraction of spin-j recoupled wires gives a factor of $2j+1$ for each vertex of the spinfoam, or, equivalently, for each point of the triangulation.
2. The normalization in the denominator of the intertwiner gives a factor of $(2j+1)^{-1}$ for each edge of the spinfoam, or, equivalently, for each segment of the triangulation.
3. The dimension of the representation d_ρ gives a factor of $2j+1$ for each face of the spinfoam, or, equivalently, for each triangle of the triangulation.

We arrive at the following formula for the partition function of the quantum BF theory:

$$Z(\Delta) = \sum_j (2j+1)^{N_p - N_s + N_t} = \sum_j (2j+1)^{\chi},$$

where N_p is the number of points in Δ, N_s is the number of segments, N_t is the number of triangles, and $\chi = N_p - N_s + N_t$ is the Euler characteristic of the triangulation, which is known to be triangulation-independent. It only depends on the genus of the surface which is triangulated.

Thus we've proven that TOCY partition function is triangulation-independent in 2 spacetime dimensions. The apparatus of Loop Quantum Gravity can be replaced with a smooth, refined background-independent quantum BF theory to give amplitudes between geometric states corresponding to global topological tegrees of freedom.

The partition function can be assigned to manifolds, not triangulations. Triangulating spacetime and using spinfoam formalism is just an intermediate step.

This is a defining feature of all topological field theories. For instance, it is possible to prove (I hope to prove it later) that BF theory is triangulation-independent in *any* number of spacetime dimensions. However, physically relevant theories like General Relativity in 4D have local degrees of freedom, and spinfoam amplitudes become dependent on the triangulation. There is still strong numerical evidence supporting the claim that the classical limit is triangulation-independent and reduces to General Relativity though.