String scattering

In this note we calculate the bosonic string scattering amplitude for the ground states (tachyons) in the critical spacetime dimension $D = 26$.

The starting point in this calculation is the Polyakov path integral over the worldsheet geometries and embeddings into the flat D-dimensional spacetime:

$$\langle \ldots \rangle = \int \frac{DhDX}{Df} \exp\left\{-\frac{1}{16\pi}\int d^2\sigma \sqrt{h(\sigma)}\, h^{ab}(\sigma)\, \partial_a X^\mu(\sigma)\partial_b X^\mu(\sigma)\right\} \cdot \ldots [X, h],$$

where the measures are by definition diffeomorphism-invariant. We could write the formal expression for the infinite-dimensional configuration space metric, from which the formal expression for the measures follows:

$$\|\delta X\|^2 = \int d^2\sigma \sqrt{h(\sigma)}\, \delta X^\mu(\sigma)\delta X^\mu(\sigma);$$

$$\|\delta h_{ab}\|^2 = \int d^2\sigma \sqrt{h(\sigma)}\, h^{ab}(\sigma)h^{cd}(\sigma)\, \delta h_{ac}(\sigma)\delta h_{bd}(\sigma).$$

We also specify the object which we wish to compute: the pin-point amplitude of the worldsheet with n spacetime points pinned to the given locations $x^\mu_{(i)}$. Such an amplitude is given by the following correlator:

$$G(x_{(1)}, \ldots, x_{(n)}) = \left\langle \prod_i \int d^2\sigma_{(i)} \sqrt{h(\sigma_{(i)})}\, \delta^{(D)}\left(x^\mu_{(i)} - X^\mu(\sigma_{(i)})\right)\right\rangle,$$

which Fourier image is given by

$$G(p_{(1)}, \ldots, p_{(n)}) = \left\langle \prod_i \int d^2\sigma_{(i)} \sqrt{h(\sigma_{(i)})}\, e^{ip^\mu_{(i)} X^\mu(\sigma_{(i)})}\right\rangle.$$

Calculating amplitudes

After fixing the gauge and dealing carefully with the conformal anomaly, the expression for the pinned string amplitudes has reduced to the following path integral:

$$G(p_{(1)}, \ldots, p_{(n)}) = \int D\varphi \exp\left\{-\frac{26-D}{48\pi}\int d^2\sigma \left[\frac{1}{2}(\partial\varphi)^2 + \mu^2 e^\varphi\right]\right\} \times$$

$$\times \prod_i \int d^2\sigma_{(i)} e^{\varphi(\sigma_{(i)})} \times \int DX_0 \exp\left\{-\frac{1}{16\pi}\int d^2\sigma\, (\partial X^\mu)^2 + ip^\mu_{(i)} X^\mu(\sigma_{(i)})\right\}.$$

Here DX_0 denotes the flat (conformal-invariant) measure, defined by

$$\|\delta X\|_0 = \int d^2\sigma\, \delta X^\mu(\sigma)\delta X^\mu(\sigma).$$

The last part is easy to compute since the DX_0 integral is Gaussian:

$$\int DX_0 \exp\left\{-\frac{1}{16\pi}\int d^2\sigma\, (\partial X^\mu)^2 + ip^\mu_{(i)} X^\mu(\sigma_{(i)})\right\} =$$

$$= \exp\left\{-\frac{1}{2}\sum_{i,j} p^\mu_{(i)} p^\mu_{(j)} \cdot \Delta\left(\sigma_{(i)}, \sigma_{(j)}\right)\right\}, \tag{1}$$

where

$$\Delta\left(\sigma_{(i)}, \sigma_{(j)}\right) = -\log\left|\sigma_{(i)} - \sigma_{(j)}\right|^2 = -2\log\left|\sigma_{(i)} - \sigma_{(j)}\right|$$

is the propagator of the differential operator $-\partial^2/8\pi$. The expression 1 is ill-defined, because it contains singular terms with $i = j$. We therefore have to regularize the resulting expression by imposing a kind of a short-distance cut-off:

$$\Delta\left(\sigma_{(i)}, \sigma_{(j)}\right) = -\log\left|\sigma_{(i)} - \sigma_{(j)}\right|^2 \to -\log\left[\left|\sigma_{(i)} - \sigma_{(j)}\right|^2 + \epsilon^2\right];\ \epsilon \to 0. \tag{2}$$

However, this cut-off is not diffeomorphism-invariant. Instead, we have to use the diffeomorphism invariant cut-off

$$\varepsilon^2 = ds^2 = h_{ab}(\sigma)\, d\sigma^a d\sigma^b = e^{\varphi(\sigma)} \cdot \delta_{ab} d\sigma^a d\sigma^b = e^{\varphi(\sigma)} \cdot \epsilon^2.$$

Substituting $\epsilon^2 = \varepsilon^2 e^{-\varphi(\sigma)}$ into 2, we get the diffeomorphism-invariant regularization

$$\Delta\left(\sigma_{(i)}, \sigma_{(j)}\right) = -\log\left[\left|\sigma_{(i)} - \sigma_{(j)}\right|^2 + \varepsilon^2 e^{-\varphi(\sigma)}\right].$$

Unless $\sigma_{(i)} = \sigma_{(j)}$, we can easily drop the ε term because at the end of the day we take the $\varepsilon \to 0$ limit. But there are contact terms in the exponent which contain

$$\Delta\left(\sigma, \sigma\right) = -\log\left[\varepsilon^2 e^{-\varphi(\sigma)}\right] = \varphi(\sigma) + \log\left[\frac{1}{\varepsilon^2}\right].$$

If we are to subtract the logarithmical divergence from 1 in a diffeo-invariant way, we are left with

$$\int DX_0 \exp\left\{-\frac{1}{16\pi}\int d^2\sigma\, (\partial X^\mu)^2 + ip^\mu_{(i)} X^\mu(\sigma_{(i)})\right\} \sim$$

$$\sim \exp\left\{-\frac{1}{2}\sum_{i\neq j} p^\mu_{(i)} p^\mu_{(j)} \cdot \Delta\left(\sigma_{(i)}, \sigma_{(j)}\right) - \frac{1}{2}\sum_i p^2_{(i)} \cdot \varphi(\sigma_{(i)})\right\} =$$

$$= \exp\left\{\sum_{i<j} p^\mu_{(i)} p^\mu_{(j)} \cdot \log\left|\sigma_{(i)} - \sigma_{(j)}\right|^2 + \frac{1}{2}\sum_i m^2_{(i)} \cdot \varphi(\sigma_{(i)})\right\} =$$

$$= \prod_{i<j} \left|\sigma_{(i)} - \sigma_{(j)}\right|^{2p^\mu_{(i)} p^\mu_{(j)}} \cdot \exp\left\{\frac{1}{2}\sum_i m^2_{(i)} \cdot \varphi(\sigma_{(i)})\right\},$$

which gives the following formula for the string scattering pin-amplitude in terms of the correlations of the Liouville theory:

$$G(p_{(1)}, \ldots, p_{(n)}) = \int D\varphi \exp\left\{-\frac{26 - D}{48\pi}\int d^2\sigma \left[\frac{1}{2}\left(\partial\varphi\right)^2 + \mu^2 e^\varphi\right]\right\} \times$$

$$\times \prod_i \int d^2\sigma_{(i)} e^{\Delta_{(i)}\varphi(\sigma_{(i)})} \times \prod_{i<j} \left|\sigma_{(i)} - \sigma_{(j)}\right|^{2p^\mu_{(i)} p^\mu_{(j)}},$$

where $\Delta_{(i)} = 1 + m^2_{(i)}/2$.

Virasoro-Shapiro amplitude

In the critical spacetime dimension $D = 26$, the Liouville theory action degenerates and the Liouville correlators exist only if

$$\forall i:\ \Delta_{(i)} = 1 + \frac{m^2_{(i)}}{2} = 0,$$

which determines the (negative) mass-squared of the closed-string tachyon in dimensionless units.

The amplitude renders

$$G(p_{(1)}, \ldots, p_{(n)}) = \prod_i \int d^2\sigma_{(i)} \times \prod_{i<j} \left|\sigma_{(i)} - \sigma_{(j)}\right|^{2p^\mu_{(i)} p^\mu_{(j)}}.$$

On the worldsheet with the topology of the sphere, this expressions enjoys $SL(2,\mathbb{C})$ conformal invariance which is remnant from the $D\varphi$ integration. We are thus free to fix any 3 of the given n points (since it is known that any 3 points can be moved onto any other 3 points with an $SL(2,\mathbb{C})$ transformation). The conventional way is to fix the three points on the Riemann sphere (complex plane with the special infinity point added):

$$\sigma_{(1)} = 0,\ \sigma_{(2)} = 1,\ \sigma_{(3)} = \infty.$$

The remaining $n - 3$ points are integrated over.

To illustrate this point, lets compute the 4-point amplitude. We start by introducing the Mandelstam variables:

$$s = \left(p_{(1)} + p_{(2)}\right)^2 = \left(p_{(3)} + p_{(4)}\right)^2,$$
$$t = \left(p_{(1)} + p_{(3)}\right)^2 = \left(p_{(2)} + p_{(4)}\right)^2,$$
$$u = \left(p_{(1)} + p_{(4)}\right)^2 = \left(p_{(2)} + p_{(3)}\right)^2.$$

The amplitude is thus equal to

$$A(s,t,u) \sim \int d^2\sigma_{(4)} \cdot \prod_{i<j} \left|\sigma_{(i)} - \sigma_{(j)}\right|^{2p^\mu_{(i)}p^\mu_{(j)}} =$$
$$= \int d^2\sigma_{(4)} \cdot \left|\sigma_{(1)} - \sigma_{(2)}\right|^{2p^\mu_{(1)}p^\mu_{(2)}} \cdot \left|\sigma_{(1)} - \sigma_{(3)}\right|^{2p^\mu_{(1)}p^\mu_{(3)}} \cdot$$
$$\cdot \left|\sigma_{(1)} - \sigma_{(4)}\right|^{2p^\mu_{(1)}p^\mu_{(4)}} \cdot \left|\sigma_{(2)} - \sigma_{(3)}\right|^{2p^\mu_{(2)}p^\mu_{(3)}} \cdot \left|\sigma_{(3)} - \sigma_{(4)}\right|^{2p^\mu_{(3)}p^\mu_{(4)}}.$$

By putting $\sigma_{(1,2,3)}$ into these specific points $(0, 1, \infty)$ we have canceled the contribution of three out of five terms. The remaining terms read

$$A(s,t,u) \sim \int d^2z\ |z|^{2p^\mu_{(1)}p^\mu_{(4)}}\ |1-z|^{2p^\mu_{(2)}p^\mu_{(4)}} \sim$$
$$\sim \frac{\Gamma\left[\frac{s}{2}-1\right]\Gamma\left[\frac{t}{2}-1\right]\Gamma\left[\frac{u}{2}-1\right]}{\Gamma\left[2-\frac{s}{2}\right]\Gamma\left[2-\frac{t}{2}\right]\Gamma\left[2-\frac{u}{2}\right]},$$

which is exactly the Virasoro-Shapiro amplitude for the closed string tachyon scattering.