SUSY Lecture #1

Poincare algebra:

$$P_\mu = i\partial_\mu,$$
$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu,$$
$$[P_\mu, P_\nu] = 0,$$
$$[P_\mu, M_{\alpha\beta}] = i\left(\eta_{\alpha\mu} P_\beta - \eta_{\beta\mu} P_\alpha\right),$$
$$[M_{\mu\nu}, M_{\alpha\beta}] = i\left(\eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha} + \eta_{\mu\beta} M_{\nu\alpha}\right),$$

SUSY algebra is extended by superchanges Q_{ia} (a — spinor index, $i \in 1 \ldots N$ — supercharge number). Let γ be Dirac gamma matrices and let C be the charge conjugation matrix, then

$$[P_\mu, Q_{ia}] = 0,$$
$$[Q_{ia}, M_{\alpha\beta}] = \frac{i}{2}\left(\gamma_{\alpha\beta}\right)_a^{\ b} Q_{ib}, \qquad \gamma^{\alpha\beta} = \frac{1}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right),$$
$$\{Q_{ia}, Q_{jb}\} = -2\delta_{ij}\left(\gamma^\mu C\right)_{ab} P_\mu.$$

SUSY algebra is a superalgebra — $\mathbb{Z}_2$-graded Lie algebra. The anticommutator should be symmetric in both (ij) and (ab) and satisfy the $\mathbb{Z}_2$-graded Jackobi identity. Symmetricity in (ab) can be checked:

$$\left(\gamma^\mu C\right)^T = C^T \gamma^{\mu T} = C^{-1}\gamma^{\mu T} C C^{-1} = -\gamma^\mu C^{-1} = \gamma^\mu C \quad \Longrightarrow \quad \left(\gamma^\mu C\right)_{ab} = \left(\gamma^\mu C\right)_{ba}.$$

The operator $P_\mu^2 = P_\mu P^\mu$ commutes with every element of the SUSY algebra — it is a Casimir operator. Thus the irreps of the SUSY algebra are labeled by an eigenvalue of P_μ^2 — the mass square m^2 of the representation.

There are two cases:

1. Massless irreps have $P_\mu^2 = m^2 = 0$.
2. Massive irreps have $P_\mu^2 = m^2 \neq 0$.

Massless SUSY irreps

We can choose the frame of reference (the basis of the representation) such that $P^\mu = \{E, 0, 0, E\}$. Massless Poincare irreps are labaled by helicity λ:

$$\lambda = \frac{\vec{P}\vec{M}}{E} = M_{12}, \quad M_i = \varepsilon_{ijk} M_{jk},$$
$$\hat{M}_{12}\left|E, \lambda\right\rangle = \lambda\left|E, \lambda\right\rangle.$$

Now lets consider supercharges:

$$\hat{Q}_{ia}\left|E, \lambda\right\rangle = ?$$

Lets see how the application of a supercharges changes the state's helicity:

$$\hat{M}_{12}\hat{Q}_{ia}\left|E, \lambda\right\rangle = [M_{12}, Q_{ia}] + Q_{ia} M_{12}\left|E, \lambda\right\rangle = -\frac{i}{2}\left(\gamma_{12}\right)_a^{\ b} Q_{ib}\left|E, \lambda\right\rangle + \lambda Q_{ia}\left|E, \lambda\right\rangle$$

$$M_{12}\begin{pmatrix} Q_{i1} \\ Q_{i2} \\ Q_{i3} \\ Q_{i4} \end{pmatrix}\left|E, \lambda\right\rangle = \begin{pmatrix} \lambda - \frac{1}{2} & 0 & 0 & 0 \\ 0 & \lambda + \frac{1}{2} & 0 & 0 \\ 0 & 0 & \lambda - \frac{1}{2} & 0 \\ 0 & 0 & 0 & \lambda + \frac{1}{2} \end{pmatrix}\begin{pmatrix} Q_{i1} \\ Q_{i2} \\ Q_{i3} \\ Q_{i4} \end{pmatrix}\left|E, \lambda\right\rangle.$$

We see that the helicity is changed by $\pm\frac{1}{2}$ after the supercharge is applied. This can not be fixed by a choice of basis or normalization. Consequently, the supercharges exchanges the bosonic and fermionic degrees of freedom.

- Q_{i1} & Q_{i3} decrease the helicity by $\frac{1}{2}$.
- Q_{i2} & Q_{i4} increase the helicity by $\frac{1}{2}$.

Now consider the anticommutator of two supercharges:

$$\{Q_{ia}, Q_{jb}\} = -2\delta_{ij} (\gamma^\mu C)_{ab} P_\mu = -2\delta_{ij} E \left[\left(\gamma^0 C\right)_{ab} - \left(\gamma^3 C\right)_{ab} \right] = 4\delta_{ij} E \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the only nontrivial anticommutator being

$$\{Q_{i2}, Q_{i3}\} = 4\delta_{ij} E.$$

Thus, Q_{i1} and Q_{i4} act as zero operators on our irrep:

$$Q_{i1} |E, \lambda\rangle = 0, \quad Q_{i4} |E, \lambda\rangle = 0.$$

Now lets put everything together. We define

$$q_i = \frac{1}{2\sqrt{E}} Q_{i3}, \quad q_i^\dagger = \frac{1}{2\sqrt{E}} Q_{i2}.$$

Btw Q_{ia} is a Majorana spinor, hence the dagger. We get the following anticommutators:

$$\{q_i, q_j\} = 0, \quad \left\{q_i^\dagger, q_j^\dagger\right\} = 0, \quad \left\{q_i, q_j^\dagger\right\} = \delta_{ij}.$$

These form a Clifford algebra. We are now seeking for irreps of the Clifford algebra.

Let λ_0 be a maximal weight of λ. Since q_i and $q_i^\dagger$ decreases and increases helicity by $\frac{1}{2}$ respectively, for the maximal weight we have:

$$q_i^\dagger |E, \lambda_0\rangle = 0.$$

Thus, we can classify the Poincare irrep into supermultiplets:

State	Helicity λ	Degeneracy of the state
$\|E, \lambda_0\rangle$	λ_0	1
$q_i \|E, \lambda_0\rangle$	$\lambda_0 - \frac{1}{2}$	N (one state for each i)
$q_i q_j \|E, \lambda_0\rangle$	$\lambda_0 - 1$	$C_N^2 = N(N-1)/2$, since $q_i q_j = -q_j q_i$
$q_i q_j \dots q_k \|E, \lambda_0\rangle$	$\lambda_0 - \frac{n}{2}$	C_N^n
$q_1 q_2 \dots q_N \|E, \lambda_0\rangle$	$\lambda_0 - \frac{N}{2}$	$1 = C_N^N$

Thus we have completed the classification of massless SUSY irreps!

Equality of the number of bosonic and fermionic degrees of freedom: Let $|E, \lambda_0\rangle$ be bosonic ($\lambda_0 \in \mathbb{Z}$). Then the number of bosonic degrees of freedom minus the number of fermionic degrees of freedom is equal to

$$C_N^0 - C_N^1 + C_N^2 - \dots = \sum_{k=0}^{N} (-1)^k C_N^k = (1-1)^N = 0,$$

where the Newton binomial has been used. Thus we have

$$n_{\text{Bose}} = n_{\text{Fermi}}, \quad \text{Q.E.D.}$$

Btw we could also compute the total number of degrees of freedom:

$$n_{\text{Bose}} + n_{\text{Fermi}} = 2n_{\text{Bose}} = 2n_{\text{Fermi}} = \sum_{k=0}^{N} C_N^k = 2^N.$$

Examples of massless SUSY reps

Example #1: $N = 1$, $\lambda_0 = 0$. We have two states:

1. $\left|E, -\frac{1}{2}\right\rangle$ — a single fermionic degree of freedom.
2. $|E, 0\rangle$ — a single bosonic degree of freedom.

It is well-known that the CPT transformation flips the sign of the helicity. Thus this irrep does not form a representation of the CPT operator, meaning that it is unphysical.

Example #2: Take a direct sum of the irrep from example #1 with $N = 1$, $\lambda_0 = \frac{1}{2}$. We get a reducible representation as a result:

1. $\left|E, -\frac{1}{2}\right\rangle_1$ — one fermion.
2. $|E, 0\rangle_1$ и $|E, 0\rangle_2$ — two bosons.
3. $\left|E, \frac{1}{2}\right\rangle_2$ — another fermion.

In future we will draw tables like this:

λ	$-\frac{1}{2}$	0	$\frac{1}{2}$
n	1	2	1

This is called the Wess-Zumino model. It consists of a complex scalar field and its superpartner — a Majorana fermion.

Example #3: Take a direct sum of the following irreps: $\lambda \in \left(-1, -\frac{1}{2}\right)$ and $\lambda \in \left(\frac{1}{2}, 1\right)$.

λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
n	1	1	0	1	1

This is the $N = 1$ supersymmetric Yang-Mills model.

Example #4: Take

λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
n	1	4	6	4	1

This is the famous (for its finiteness) $N = 4$ supersymetric Yang-Mills model.

Example #5: Take

λ	-2	$-\frac{3}{2}$
n	1	1

$+$

λ	$\frac{3}{2}$	2
n	1	1

This is the $N = 1$ supergravity theory. In its spectrum there are spin-2 gravitons and their superpartners — spin-$\frac{3}{2}$ gravitinos.

Example #6: Take

λ	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
n	1	8	C_8^2	C_8^3	C_8^4	C_8^3	C_8^2	8	1

This is the $N = 8$ supergravity theory.

Fact: no noncontroversial Lagrangian dynamics exists for spin-$\frac{5}{2}$ and greater. Thus, we can only increase N up to 8 in 4-dimensional spacetime.

Massive SUSY irreps

Now $P_\mu^2 = m^2 \neq 0$. Choose the stationary frame of reference: $P_\mu = \{m, 0, 0, 0\}$. The massive Poincare irreps are labeled by the square of the angular momentum $S(S+1)$ and its projection on the OZ axis:

$$\hat{M}_{12} |S, S_3\rangle = S_3 |S, S_3\rangle, \quad \vec{M}^2 |S, S_3\rangle = S(S+1) |S, S_3\rangle.$$

Just like in the massless case, we consider the supercharges:

$$M_{12} Q_{ia} |S, S_3\rangle = \begin{pmatrix} S_3 - \frac{1}{2} & 0 & 0 & 0 \\ 0 & S_3 + \frac{1}{2} & 0 & 0 \\ 0 & 0 & S_3 - \frac{1}{2} & 0 \\ 0 & 0 & 0 & S_3 + \frac{1}{2} \end{pmatrix} Q_{ia} |S, S_3\rangle.$$

We conclude that Q_{i1} & Q_{i3} decrease S_3 by $\frac{1}{2}$, while Q_{i2} & Q_{i4} do the direct opposite. Now lets calculate the anticommutator:

$$\{Q_{ia}, Q_{jb}\} = -2\delta_{ij}\left(\gamma^\mu C\right)_{ab} P_\mu = 2m\delta_{ij}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

which is somewhat different from the analogous result in the massless case. We introduce a capital index $I \in 1\dots 2N$ and generators

$$q_I = \begin{cases} \frac{1}{\sqrt{2m}} Q_{I3} & I \in 1\dots N, \\ \frac{1}{\sqrt{2m}} Q_{(I-N),1} & I \in N+1\dots 2N, \end{cases}$$

$$q_I^\dagger = \begin{cases} \frac{1}{\sqrt{2m}} Q_{I2} & I \in 1\dots N, \\ \frac{1}{\sqrt{2m}} Q_{(I-N),4} & I \in N+1\dots 2N. \end{cases}$$

These form a $2N$-dimensional Clifford algebra:

$$\{q_I, q_J\} = 0, \quad \left\{q_I^\dagger, q_J^\dagger\right\} = 0, \quad \left\{q_I, q_J^\dagger\right\} = \delta_{IJ}.$$

Now q_I and $q_I^\dagger$ decrease and increase S_3 by $\frac{1}{2}$ respectively. Like in the massless case, we classify the Poincare irrep into supermultiplets:

State	Spin projection S_3	State degeneracy
$\lvert S_3^{\max}\rangle$	S_0	1
$q_I \lvert S_3^{\max}\rangle$	$S_0 - \frac{1}{2}$	$2N$ (one for each I)
$q_I q_J \lvert S_3^{\max}\rangle$	$S_0 - 1$	$C_{2N}^2 = 2N(2N-1)/2$, since $q_I q_J = -q_J q_I$
$q_I q_J \dots q_K \lvert S_3^{\max}\rangle$	$S_0 - \frac{n}{2}$	C_{2N}^m
$q_1 q_2 \dots q_{2N} \lvert S_3^{\max}\rangle$	$\lambda_0 - N$	$1 = C_{2N}^{2N}$

The following aspects are different in the massive case:

1. Unlike helicity, S_3 only takes the following values: $S_3 \in -S_0 \dots S_0$. This means that

$$S_0 - N = -S_0 \quad \Longrightarrow \quad S_0 = \frac{N}{2}.$$

2. S also takes integer values.

$S\backslash N$	1	2	3	4
0	2	$C_4^2 - 1 = 5$	14	42
$\frac{1}{2}$	1	4	14	48
1	-	1	6	27
$\frac{3}{2}$	-	-	1	8
2	-	-	-	1