The leptonic sector of SM

The leptonic sector of SM consists of three generations:

$SU(2)$ rep.	Hypercharge, Y	1-st generation, $I=1$	2nd generation, $I=2$	3rd generation, $I=3$
fundamental, $\sigma^a/2$	$-1/2$	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$
trivial	-1	e_R	μ_R	τ_R
trivial	0	$\nu_{e,R}$	$\nu_{\mu,R}$	$\nu_{\tau,R}$

The right-handed neutrinos have been hypothetical for a long time, but nowadays compelling evidence in favor of their existence has formed.

The kinematic part of the Lagrangian of the leptonic sector (we don't include the right-handed neutrinos) is equal to

$$\mathcal{L}_{\text{kinematic}} = i\overline{\begin{pmatrix} \nu & e \end{pmatrix}}^I_L \gamma^\mu D_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L + i\bar{e}^I_R \gamma^\mu D_\mu e^I_R.$$

Mass terms are incompatible with the $SU(2)$ invariance, but we can replace them with the gauge-invariant Yukawa interactions with the Higgs field:

$$\mathcal{L}_{\text{mass-Yukawa}} = -\left(Y_e\right)_{IJ} \overline{\begin{pmatrix} \nu & e \end{pmatrix}}^I_L \begin{pmatrix} \varphi_1 + i\varphi_2 \\ v + \varphi_3 + i\varphi_4 \end{pmatrix} e^J_R + \text{c.c.}$$

The covariant derivatives act on the left-handed doublet and the right-handed singlet through

$$D_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L = \partial_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L + \underset{SU(2)}{A_\mu} \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L - \frac{ie_1}{2} \underset{U(1)}{A_\mu} \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L,$$

$$D_\mu e_R = \partial_\mu e_R - ie_1 \underset{U(1)}{A_\mu} e_R.$$

The overall Lagrangian is thus

$$\mathcal{L}_{\text{leptons}} = \mathcal{L}_{\text{kinematic}} + \mathcal{L}_{\text{mass-Yukawa}}.$$

We now check the $SU(3) \times SU(2) \times U(1)$ invariance of this Lagrangian:

1. The $SU(3)$ invariance is obvious (since all leptons belong to the trivial representation of $SU(3)$).
2. The $SU(2)$ invariance is satisfied because we use covariant derivatives instead of ordinary partial derivatives.
3. The $U(1)$ invariance is satisfied because in each term of the overall Lagrangian the sum of all hypercharges is equal to 0. From the expression for the covariant derivative, we have $Y_{e_L} = -1/2$, $Y_{e_R} = -1$, $Y_\phi = +1/2$. Thus, the sum of hypercharges for the mass term is $-Y_{e_L} + Y_\phi + Y_{e_R} = 0$ and it is $U(1)$-invariant.

Now we investigate what the leptonic Lagrangian turns into in the low-energy limit. The masses of $W^\pm$ and Z^0 are huge, so these particles are absent in the limit of low energies. We are left with:

$$D_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L = \partial_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L + \underset{SU(2)}{A_\mu} \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L - \frac{ie_1}{2} \underset{U(1)}{A_\mu} \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L = \partial_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L +$$

$$\frac{ie_2}{2}\left(W^1_\mu \sigma^1 + W^2_\mu \sigma^2\right) \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L + \frac{ie_2}{2}\left(Z_\mu \cos\theta_W + A_\mu \sin\theta_W\right)\sigma^3 \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L -$$

$$-\frac{ie_1}{2}\left(A_\mu \cos\theta_W - Z_\mu \sin\theta_W\right) \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L \to \partial_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L + \frac{ie}{2}\left[\sigma^3 - 1\right] A_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L =$$

$$= \partial_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L - ie \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A_\mu \begin{pmatrix} \nu \\ e \end{pmatrix}^I_L,$$

$$D_\mu e^I_R = \partial_\mu e^I_R - ie_1\left(A_\mu \cos\theta_W - Z_\mu \sin\theta_W\right) e_R \to$$

$$\rightarrow \partial_\mu e_R^I - ieA_\mu e_R.$$

Remember that $e = e_1 \cos\theta_W = e_2 \sin\theta_W$. We see that in the low energy limit, where only the massless photon survives, neutrinos become neutral whereas both left-handed and right-handed electrons acquire electromagnetic charge. The symmetry between left-handed and right-handed particles recovers.

We now pass on to the diagonalization of the matrix $(Y_e)_{IJ}$.

Theorem: Let Y be a nondegenerate $n \times n$ matrix. Then there exist two unitary $n \times n$ matrices A and B such that

$$D = A^{-1} Y B$$

is diagonal, real-valued and positive-definite.

Proof: Consider $YY^\dagger$. It is obviously self-adjoint since $\left(YY^\dagger\right)^\dagger = Y^{\dagger\dagger}Y^\dagger = YY^\dagger$. Thus, there exists a unitary matrix A such that

$$D^2 = A^{-1} Y Y^\dagger A$$

is diagonal, real-valued (and thus self-adjoint) and positive-definite. These conditions are enough to define a square root matrix

$$D = \sqrt{D^2}.$$

We choose $B = Y^{-1}AD$ to be our B matrix. It is unitary:

$$\left(B^{-1}\right)\left(B^{-1}\right)^\dagger = D^{-1}A^{-1}YY^\dagger\left(A^{-1}\right)^\dagger\left(D^{-1}\right)^\dagger = 1.$$

A direct calculation shows that

$$A^{-1}YB = A^{-1}YY^{-1}AD = D,$$

which, as we have already seen, is diagonal, real-valued and positive-defined.

This theorem allows us to choose $(Y_e)_{IJ}$ to be diagonal, real-valued and positive-definite by means of a unitary transformation. Because this transformation is unitary, it doesn't affect the kinetic term in the Lagrangian. Thus, we see that the Yukawa interaction term reduces to masses of three generations of leptons in the low-energy limit and all cross-terms are absent. The masses of leptons become

$$\begin{cases} m_e = v \cdot (Y_e)_{11}\,, & m_{\nu_e} = 0 \\ m_\mu = v \cdot (Y_e)_{22}\,, & m_{\nu_\mu} = 0 \\ m_\tau = v \cdot (Y_e)_{33}\,, & m_{\nu_\tau} = 0 \end{cases}$$

Fact: the numeric values of the masses of leptons satisfy the following relation with great accuracy:

$$\frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{m_e + m_\mu + m_\tau}} \approx \sqrt{\frac{3}{2}}.$$

No one knows why this is so.