Spin networks in LQG

In canonical quantum gravity, we deal with three kinds of constraints:

1. The Gauss constraint ($G_i = D_a E_i^a$ in Ashtekar-Barbero variables) fixes the internal $SU(2)$ freedom in the choice of new canonical variables. It has a natural interpretation in terms of the Einstein-Hamilton-Jacobi formalism: $G_i = 0$ is equivalent to the requirement for the Hamilton's functional $S[A]$ (as a functional of the Asktekar-Barbero connection) to be gauge-invariant.

2. The diffeomorphism constraint ($F_a = F_{ab}^i E_i^b$) fixes the 3d diffeomorphism invariance. In terms of the Einstein-Hamilton-Jacobi formalism $F_a = 0$ is equivalent to the requirement for $S[A]$ to be diffeomorphism-invariant.

3. The Hamiltonian constraint (aka Wheeler-DeWitt equation) is supposed to fix the remaining degrees of freedom of the 4d diffeomorphism invariance. Although, in quantum theory we have to be careful with this statement as it can in theory be spoiled by quantum anomalies which could arise in the hypersurface deformation algebra. In Ashekar-Barbero variables:
$$H = \left(\gamma^2 F_{ab}^i + \left(\gamma^2 - \frac{s}{4}\right) \varepsilon_{imn} K_a^m K_b^n\right) \cdot \frac{\varepsilon^{ikl} E_k^a E_l^b}{\sqrt{\det q}},$$
where γ is the Immirzi parameter and $s = -1$ is the signature of the metric (the choice $s = -1$ corresponds to the Lorentzian physical theory, while $s = 1$ corresponds to the Euclidean unphysical theory). The pure imaginary choice for the Immirzi parameter ($\gamma = i/2$) corresponds to the Ashtekar connection, and simplifies dramatically the expression for the Hamiltonian constraint. However, it does so at the cost of introducing a nonpolynomial reality constraints (which ensure that the physical metric q_{ab} is real). Consequently, the modern treatment is to make γ real.

Unlike the Wheeler-deWitt constraint, the first two constraints admit a concise set of solutions in the quantum theory. We have a natural ordering for the quantum operators $\hat{G}_i$ and $\hat{F}_a$: these ensure the wavefunctional $\Psi[A]$ to be gauge-invariant and 3d diffeomorphism-invariant respectively. This is a natural generalization of the Einstein-Hilbert-Jacobi interpretation to the quantum theory.

Surprisingly, the space $\mathcal{K}$ of these solutions admits a countable basis, which is a necessary condition for it to be a Hilbert space.

Graph discretization

We start by considering a path $x : [0..1] \to \Sigma$ in the 3d spatial manifold. In the presence of the $SU(2)$ gauge connection A (Ashtekar-Barbero connection), an important geometrical object is associated to x, which is the holonomy functional evaluated on x:
$$h_x[A] = \mathrm{P}\exp \int_0^1 A_a(x(\tau))\, \dot{x}^a(\tau)\, d\tau \ \in \ SU(2).$$
The holonomy is obviously a classical quantity: it depends on a particular classical configuration $A_a(x)$.

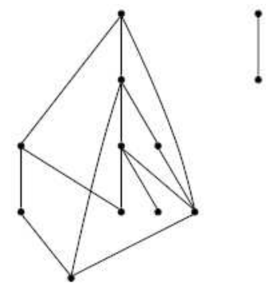
Our objective is to pass from a description of the system given in terms of the connection A to another, given in terms of the holonomies of paths. Intuitively we know that holonomies are much more fundamental than the connections themselves. For example, the holonomies continue to make sense in the extreme cases, when the smooth connection can not be defined.

Ofcourse, if we choose the holonomy functional $u_x = h_x[A]$ to parametrize the classical system instead of A, we loose all the information contained in A about the classical configuration away from the path x.

The solution to this problem is obvious: we keep introducing new paths until all regions of interest are filled with paths and their endpoints. We end up with a collection of paths $\{x_i\}$ and their holonomies $\{u_i = h_{x_i}[A]\}$, which we happily substitute for the connection function $A_a(x)$ in the classical description of the system.

In fact we can generalize this further to the case when different paths can meet at their endpoints. The resulting structure is called an oriented graph, embedded into Σ. The endpoints of paths are called the *nodes* of the graph, whilst the paths themselves are called *links*.

The following figure shows an example of the graph which we consider here. This particular example contains an isolated link, which forms the second connected component of the graph:



To each link e of the graph we associate a group element $u_e \in SU(2)$, which is by definition equal to the holonomy of A evaluated on the embedding path of this link. We expect the graph to be complex enough in order to capture as much information about A as needed to solve any particular problem.

It might seem that the theory depends on the particular choice of the graph Γ, which is unelegant. This is indeed true for now, but, as will be shown later, in the quantum theory diffeomorphism invariance forces us to consider a graph-independent space of quantum states. There is thus a unique Hilbert space. I actually consider this to be one of the most beautiful insights of LQG.

Quantization

In the connection formalism, quantization is achieved by passing from the classical degrees of freedom $A_a(x)$ to their quantized version, given by the wavefunctional $\Psi[A]$. Then we promote $A_a(x)$ and its conjugate momentum $E^a(x)$ to quantum operators, acting on the wavefunctional via

$$\hat{A}_a(x)\,\Psi[A] = A_a(x)\Psi[A],$$
$$\hat{E}^a(x)\,\Psi[A] = -i\hbar\frac{\delta\Psi[A]}{\delta A_a(x)}.$$

The quantum dynamics of gravity is captured in the constraints, which we have to promote to quantum operators. Note that there could be (and there are) different choices of ordering, which in general give rise to unequivalent quantum theories. Quantization is ambiguous.

In general, it is not possible to solve the constraints and give a well-defined Hilbert space structure to the space of solutions of the constraints. But instead of quantizing, we could pass to the graph-discretized degrees of freedom $\{u_e\}$ and then quantize. This turns out to give a good definition of the Hilbert space $\mathcal{K}_0$.

So we are given a graph embedded into Σ, and the classical degrees of freedom are the holonomies $\{u_e\}$ associated to the links of the graph. Canonical quantization requires us to pass to the wavefunction description. The quantum state is given by a function $\Psi(u_1, \ldots, u_m)$ of m group elements (m is the number of links in the graph).

Each argument of the wavefunction is an element of $SU(2)$. Thus, Ψ can be seen as a function

$$\Psi \in \mathcal{K}_0 : \quad SU(2)^m \to \mathbb{C}.$$

The next step is to define the quantum operators acting on the linear vector space of such wavefunctions. According to the canonical quantization, the "position" operators are the $SU(2)$-valued holonomy operators associated with links:

$$\hat{U}_e\,\Psi(u) = u_e\Psi(u).$$

Note that we don't have a connection operator $\hat{A}_a(x)$ anymore. Instead, the holonomies evaluated on the links of the graph become quantum operators. It is fundamentally impossible to obtain more information about $A_a(x)$ then it is contained in $\{u_e\}$.

The analogue of the conjugate momentum is given by the left-invariant vector fields on $SU(2)$:

$$\hat{J}^i_e\,\Psi(u) = -i\hbar\,\frac{d}{dt}\Psi\left(u_1, \ldots, u_e e^{t\sigma^i}, \ldots, u_m\right)\bigg|_{t=0}.$$

Moreover, we have a well-defined Hilbert space structure, given by the Haar measure on $SU(2)$:

$$\langle\Phi|\Psi\rangle = \prod_{e=1}^{m}\left(\int du_e\right)\;\Phi^*(u)\,\Psi(u).$$

Since $SU(2)$ is compact, we are to expect a countable basis in $\mathcal{K}_0$. Now that we have our Hilbert space, we are ready to promote the constraints to quantum operators and solve them.

Harmonic analysis on the group

As I have mentioned earlier, we have a natural definition of the Gauss and diffeomorphism constraints in the quantum theory. They can be translated as requirements for the wavefunction to be gauge-invariant and diffeomorphism-invariant. So we are left with the following problem: what are the gauge-invariant and diffeomorphism-invariant subspaces of $\mathcal{K}_0$?

In this section we consider the Gauss constraint, which is responsible for gauge invariance. We will build the space $\mathcal{K} \subset \mathcal{K}_0$ of solutions of the quantum Gauss constraint.

The first step in the way to the definition of $\mathcal{K}$ is to give an explicit countable basis in $\mathcal{K}_0$. This can be done by means of the Peter-Weyl theorem (an important result of the harmonic analysis on groups).

The Peter-Weyl theorem: consider a compact semisimple Lie group G. We are interested in the space $L_2(G)$ of complex-valued square-integrable functions over G, where we use the Haar measure to define the integrals. An orthonormal basis in $L_2(G)$ is given by a sequence of subspaces labeled by irreducible representations of the Lie algebra Lie(G):

$$L_2(G) = \bigoplus_{\rho} H_\rho,$$

where ρ labels the irreducible representations and the dimensionality of H_ρ is given by

$$\dim H_\rho = (\dim \rho)^2 .$$

The basis vectors in $L_2(G)$ are thus given by a set of Wigner functions

$$[W_\rho]_\alpha^{\ \beta} : \quad G \to \mathbb{C},$$

where $\alpha, \beta \in [1..\dim \rho]$. Moreover, α and β transform as covariant and contravariant (respectively) indices in the irreducible representation ρ.

As a warmup, consider an exceptionaly easy case: the group $G = U(1)$. It is not exactly semisimple, but the theorem applies nevertheless. The irreducibles of $U(1)$ are of topological nature. They are labeled by integer *winding numbers*. All the irreducibles of $U(1)$ are of dimension 1, so there will be no α and β indices in this case. In the polar coordinate, the Wigner functions are given by

$$W_n(\varphi) = e^{in\varphi},$$

where n is the winding number labeling the irrep. One recovers the Fourier series, which indeed gives a countable basis over $L_2(S^1)$. Thus, the Fourier series is a special case of the Peter-Weyl basis.

In our case, the group is $SU(2)^m$. The irreducible representations of $su(2)$ are labeled by nonnegative half-integers — *spins*. The dimensionality of the irreducible representation of spin j is

$$\dim j = 2j + 1.$$

Thus, the irreducibles of Lie($SU(2)^m$) are labeled with collections of spins, one spin j_e for each link e. We also have two indices α_e and β_e belonging to the spin-j_e representation for each link e. All together, we label each link with an ordered triple (j_e, α_e, β_e):

- $j_e \in \{0,\ 1/2,\ 1,\ 3/2,\ \ldots\}$,
- $1 \le \alpha_e, \beta_e \in \mathbb{N} \le \dim j_e = 2j_e + 1$,
- α_e and β_e are covariant and contravariant (respectively) indices in the spin-j_e irrep of $su(2)$.

Each allowed coloring of the graph is an element of the orthonormal basis in $L_2(SU(2)^m) = \mathcal{K}_0$. Thus, any quantum state $\Psi \in \mathcal{K}_0$ can be expanded in a sum over this basis. We say that any quantum state Ψ is a *quantum superposition* of the allowed colorings of the graph.

Each basis vector is given by an $SU(2)^m$ Wiegner function, which is just a product of the well-known $SU(2)$ Wiegner functions $[W_j]_\alpha^{\ \beta}$:

$$\Psi_{j_e,\alpha_e,\beta_e}(u) = \prod_e [W_{j_e}]_{\alpha_e}^{\ \beta_e}(u_e) .$$

Quantum states from $\mathcal{K}_0$ are superpositions of $SU(2)^m$ Wiegner functions:

$$\Psi(u) = \sum_{\{j_e,\alpha_e,\beta_e\}} C_{j_e,\alpha_e,\beta_e} \cdot \Psi_{j_e,\alpha_e,\beta_e}(u).$$

Spin networks

Now that we've gotten our hands on an orthonormal basis in $\mathcal{K}_0$, we are finally ready to express the Gauss constraint in terms of the new basis, and then solve it. The Gauss constraint requires us to only consider gauge-invariant states. Thus, we have to derive the transformation properties of the wavefunctions under gauge transformations.

Consider a gauge transformation acting on the smooth-connection description. The Ashtekar connection transforms according to

$$\underline{A}(x) \to g(x)\underline{A}(x)g^{-1}(x) + g(x)dg^{-1}(x).$$

But we are dealing with the holonomies of A evaluated on specific paths, which have a much more simple transformation properties. In fact,

$$h_x[A] \to g(s)h_x[A]g^{-1}(t),$$

where $s = x(0)$ and $t = x(1)$ are the two endpoints of the path x. We say that the holonomy is a two-point gauge tensor.

Evidently, the gauge transformation of the description of our system in terms of holonomies is parametrized by group elements $g(x)$ given at the endpoints of all the paths, which are located at the nodes of the graph. In the discretized description, we have a group element g_v associated to each node v of the graph. The information about $g(x)$ at another points of Σ is lost.

The holonomies $\{u_e\}$ transform under the gauge transformations according to

$$u_e \to g_s u_e g_t^{-1},$$

where $s(e)$ and $t(e)$ are the source and target nodes of the edge e. These are given by the abstract topology of the graph.

Consequently, the wavefunction transforms according to

$$\Psi\left(\ldots, u_e, \ldots\right) \to \Psi\left(\ldots, g_s u_e g_t^{-1}, \ldots\right).$$

Consider an element of the countable orthonormal basis defined in the previous section. It is given by a coloring of links with an ordered triple (j_e, α_e, β_e). Consider a gauge transformation, which is only nontrivial at a single node v of the graph, and equals to the group identity at all other nodes.

From the point of view of node v, the wavefunction contains a Wiegner function for any outgoing and incoming link e. According to the Peter-Weyl theorem, the Wiegner function's indices α_e and β_e transform as contravariant and covariant indices in the spin-j_e irrep. Thus we have a covariant index for any outgoing link and a contravariant index for any incoming link.

Now we would like to find linear superpositions in $\mathcal{K}_0$ which don't change under gauge transformations.

A well-defined combinatorical problem is presented to us: given a set of a incoming and b outgoing links and their spins, compute a space of invariant tensors of rank (a, b) with indices in irreps of $su(2)$ given by the spins. Each covariant index is then contracted with the contravariant index of the corresponding link, and vice versa, giving rise to gauge-invariant wavefunctions. These invariant tensors are called the group *intertwiners.*

We are now ready to describe the spin network basis in the gauge-invariant subspace $\mathcal{K} \subset \mathcal{K}_0$. Or, in another words, we've solved the Gauss constraint of tetradic General Relativity.

The spin network basis consists of the colorings of the graph, where:

- each link is colored with an irreducible representation of the gauge group ($SU(2)$ in case of gravity in Ashtekar variables),
- each node is colored with an element of any basis of the linear vector space of intertwiners of $SU(2)$ with the spins given by the colorings of incoming and outgoing edges.

The wavefunction of the spin network state is given by

$$\Psi_{\text{spin network}}(u) = \left\{\prod_e \left[W_{j_e}\right]_{\alpha_e}^{\ \beta_e}\right\} \cdot \left\{\prod_v \mathcal{I}^{\alpha_1\alpha_2\ldots}_{\beta_1\beta_2\ldots}\right\},$$

where the pattern of contraction is dictated by the abstract topology of the graph.

The space of spin networks can be evaulated for all values of spins by means of $SU(2)$ recoupling theory. Here I will only touch this briefly.

First of all, the orientation of edges is not important. We can always choose an arbitrary orientation by joggling the indices of intertwiners with the metric tensor in the spin-j irrep. It is equivalent to switching to a conjugate irrep, which in the case of $SU(2)$ does not differ from the original.

Let us compute the space of intertwiners for 3-valent nodes with all three links chosen to be incoming. These are of form $\mathcal{I}_{\beta_1\beta_2\beta_3}$ with β_1, β_2 and β_3 in irreps with spins j_1, j_2 and j_3 respectively.

In the representation theory of $su(2)$ there is an important identity:

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2| + 1 \oplus \ldots \oplus (j_1 + j_2)\,.$$

All the irreps in the range of $|j_1 - j_2| \,..\, (j_1 + j_2)$ enter this sum exactly once. Thus there is a unique (normalized) intertwiner $\mathcal{I}_{\beta_1\beta_2\beta_3}$ when j_1, j_2 and j_3 satisfy the following constraints:

1. $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$,
2. $j_1 + j_2 + j_3 \in \mathbb{Z}$.

Thus, for the 3-valent nodes we can omit the intertwiners provided that the spins of incident links satisfy the the above conditions.

S-knots and states of quantum gravity

The spin network basis consists of quantum states which are by construction solutions to the Gauss constraint. The remaining two constraints are the diffeomorphism constraints and the Wheeler-deWitt equation. In this section we solve the former.

First, we will have to develop a slightly different notion of the spin network. In the previous sections, we have considered colorings of a predefined graph Γ. This can get tricky as the theory seems to depend on a particular choice of graph Γ, which seems unelegant and requires a predefined information about the structure of Γ which is obviously unavailable for the observer.

Instead, we will consider not just a graph, but a very specific graph $\Gamma = K_\infty$. This is a "limit" of the sequence of graphs K_n, where K_n is a full graph with n nodes and a countable infinity of links joining any two nodes. Thus, $\Gamma = K_\infty$ contains a countable infinity of nodes each two being connected by a countable infinity of links.

Now consider the trivial irrep of $SU(2)$ — the spin-0 irrep. We will assign a special meaning for spin network links colored with $j_e = 0$, which is — we will declare them non-existent. The very notion of whether the link exists or not is thus also quantum — the system can be in a superposition of states, and the link can be existent and non-existent just as the Schrodinger's cat is dead and alive at the same time.

For convenience we will conspirate not to draw the non-existent links at all. Thus, we are left with the basis of all possible finite graphs (they all are subgraphs of $\Gamma = K_\infty$), colored with *faithful* irreps and intertwiners. And by j_e being faithful we mean ofcourse that $j_e \neq 0$.

Now consider an arbitrary 3d diffeomorphism. We are only interested in the diffeomorphisms lying in the connected to the group identity component of the diffeomorphism group. These are homotopies of coordinatizations of Σ.

How does the gauge connection A transform? Being an 1-form, it satisfies the tensorial transformation law:

$$A_\mu(x) \to A_\mu(y) = \frac{\partial x^\nu}{\partial y^\mu} \cdot A_\nu(x).$$

How does the holonomy $h_x[A]$ change under a diffeomorphism? It doesn't! The diffeomorphism changes both the gauge connection *and* the path x so the holonomy operators remains intact. The arguments u_e of the wavefunction $\Psi(u)$ don't change under diffeomorphisms. By construction, the spin network basis is diffeomorphism-invariant.

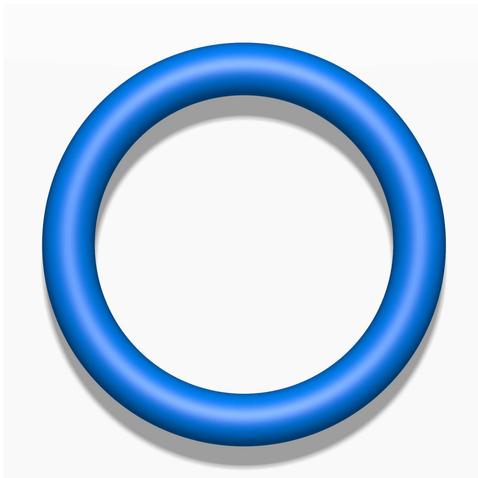
However, there's two subtleties.

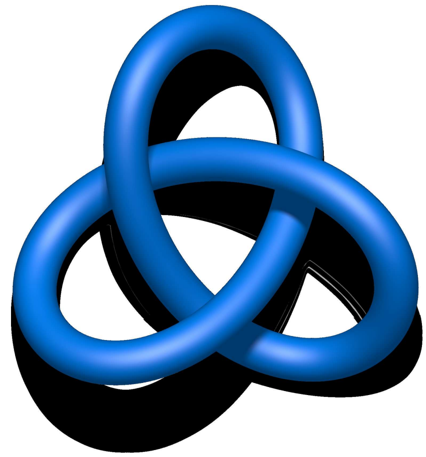
1. Diffeomorphisms can exchange the nodes of the graph. We will thus have to identify the colorings of the graph related by a relabeling of the nodes.
2. Diffeomorphisms can not make a link transit through another link. We are thus forced to treat graphs with the same abstract topology as different if the topology of their embeddings in Σ cannot be connected with a homotopy.

We are lead to the new concept: we call an *s-knot* or a *spin knot* a homotopy class of embeddings of a spin network in Σ, with the ones related by a node reordering identified as the same s-knot.

In LQG, s-knots form a basis in the kinematical space of states of quantum gravity $\mathcal{K}_s$.

Consider, for example, a special case of s-knots: ordinary knots aka homotopy classes of the embedding $S^1 \to \mathbb{R}^3$. These have been classified and labeled by knot invariants, the best known knots being the *unknot* and the two *trefoil knots* (the mirror images of one another):





The embeddings of knots in Σ are called *loops*, and the corresponding spin network state can be easily computed. Consider a loop with spin-j link e and a single node v which serves as both the source and the target of e. There is a unique (normalized) intertwiner $\mathcal{I}^\alpha_\beta = \delta^\alpha_\beta$ and the spin network wavefunction becomes

$$\Psi(u) = \delta^\alpha_\beta \left[W_j\right]_\alpha^{\ \beta}(u) = \mathrm{tr}U,$$

where U is the spin-j image of the group element u. This is the famous Wilson loop functional, or the holonomy trace functional.

Overall, the loop state is parametrized by knot invariants and the spin j of the loop. We could also have multiloop states with several (possibly tangled) loops.

Loop and multiloop states are a particularly interesting subset of s-knot states. This is because they are also solutions of the Wheeler-deWitt equations, thus being the true physical states of quantum gravity. However, these are unphysical since the volume operator vanishes on nonintersecting loops and thus the loop states describe a Universe with no space. Quantum states of the Big Bang singularity anybody?

Quantum geometry

My longstanding dream was to play with a toy model (at least) which would provide a quantum notion of the geometry of spacetime just as General Relativity provides a classical notion of geometry. With LQG I finally have this come true.

In this section we consider a 3d General Relativity (2d space + time) in frame-connection variables. It has the same kind of Gauss and diffeomorphism constraints, so the spin network basis works perfectly. However, there are no s-knots in 2 dimensions, just as there are no knots. Instead, the spin networks themselves span the kinematical space of quantum gravity in 3d provided that we identify the colorings related by graph isomorphisms.

Consider a path x in Σ. In classical theory, we could define the length of the path by the reparametrization-invariant integral

$$L_x = \int_0^1 dt\, \sqrt{q_{ab}(x(t))\, \dot{x}^a(t)\, \dot{x}^b(t)} = \int_0^1 dt\, \sqrt{e^i_a(x) e^i_b(x)\, \dot{x}^a(t)\, \dot{x}^b(t)}.$$

What comes next is an incredibly beautiful insight in how the geometry manifests itself on the quantum level.

Since the frame $e_a^i(x)$ becomes a quantum operator, we can (formally, for now) define the *length operator* by simply putting hats over L and e:

$$\hat{L}_x = \int_0^1 dt\,\sqrt{\hat{e}_a^i(x)\hat{e}_b^i(x)\,\dot{x}^a(t)\,\dot{x}^b(t)}.$$

Note that the length of the path x is not predefined! In fact, it is completely unknown unless an eigenstate of $\hat{L}_x$ is specified. So x could be any path, from the tiniest path of order of the Planck's length to the giant cosmological path connecting galaxy superclusters. The length operator is the same for all paths. What truly makes them different is the specific state of quantum gravity in which the Universe is found by the observer!

According to canonical quantization of 3d General Relativity, $\hat{e}_a^i$ acts as a derivative on wavefunctionals in the connection representation:

$$\hat{e}_a^i\Psi[\omega] = -i\cdot 8\pi\hbar G\cdot\frac{\delta\Psi[\omega]}{\delta\omega_i^a(x)}.$$

This can be used to compute the action of $\hat{e}$ on the spin network state. The spin network state depends on the holonomies of links, which themselves depend on ω. We are thus calculating

$$\frac{\delta\Psi(u(\omega))}{\delta\omega_i^a(x)} = \hat{J}_e\Psi(u)\cdot\frac{\delta u_e[\omega]}{\delta\omega_i^a(x)}.$$

The variation in the r.h.s. was computed in my previous note. It gives

$$\frac{\delta u_e[\omega]}{\delta\omega_i^a(x)} = i\int_0^1 dt\,\delta^{(2)}(x;x(t))\,\dot{x}^a(t)\cdot h_{x_2}[\omega]\sigma_a h_{x_1}[\omega],$$

where x_1 and x_2 are the two pieces of the path $x(t)$ separated by point x.

The operator $\hat{L}_x$ is still ill-defined, though.

Lets first consider the simplest possible case: x does not intersect any spin network links. In this case it is easy to see that all variations of the holonomy are equal to 0 because of the vanishing of delta functions $\delta^{(2)}(x;x(t))$. Thus, in this case the physical length of the path is zero!

The next case is the one with a single intersection of a single spin network link with spin j. This will give σ_a in the spin-j irrep when acting on a spin network with e, and thus

$$\hat{L}_x\Psi = 8\pi\hbar G\sqrt{\mathrm{tr}_j\sigma_a\sigma_a}\,\Psi = 8\pi\hbar G\sqrt{j(j+1)}\,\Psi,$$

which is exactly the Casimir of $su(2)$ in the spin-j irrep.

Now we want to know what $\hat{L}_x\Psi$ is equal to when x has multiple intersections with the spin network. The math is pretty complex and the resulting expression does not have any simple interpretation.

However, we have to remember that we are dealing with quantum theory, and there are ordering ambiguities. Thus we should define the operator $\hat{L}_x$ first. We take the following definition: in order to compute $\hat{L}_x$, split x into tiny (in the mathematical sense, remember that we don't have an apriori notion of physical length?) pieces such that each piece intersects the spin network at most once. Then add the tiny portions together (this is actually a pretty intuitive definition, since in classical theory length is additive and I see no reason why this shouldn't hold in quantum theory).

The result is the following: the length of a path x in quantum theory is represented by an operator defined on spin network states as a sum over the spin network links e which intersect x, each link being taken once for each of its intersections:

$$\hat{L}_x\,|\Psi\rangle = 8\pi\hbar G\cdot\sum_e\sqrt{j_e\,(j_e+1)}\,|\Psi\rangle\,.$$

What does this formula tell us?

1. Spin network states have a nice geometrical interpretations: they are eigenstates of quantum geometry (the length operator)!
2. There exists a minimal possible length associated with a single intersection with a spin-1/2 link. It is equal to

   $$L_{\min} = 4\sqrt{3}\pi\cdot\hbar G \approx 21.7\times L_{\text{Planck}},$$

   as in 2d the Planck length is given by $L_{\text{Planck, 2d}} = \hbar G$. Length is fundamentally quantized!

3. Note that length quantization *does not* go against Lorentz invariance. It is the *eigenvalues* that are quantized, there is still room for the continuous contractions as operators on $\mathcal{K}$.
4. In 4d we would interpret the intersections of spin networks with test surfaces as quanta of area, therefore it is area and not length that is quantized. This has far reaching consequences, including the explanation of the black hole enthropy by LQG.

Finally, consider a trivalent node of the spin network. It is associated with a chunk of space — a quantum triangle, with the length of its sides given by the formula above. These lengths depend on the three spins of the links incident to the node.

Now consider a classical regime: both three spins are extremely large. We have an approximate relation

$$\hat{L}_x |\Psi\rangle \sim j_e |\Psi\rangle ,$$

since the +1 in the square root does not matter much. So the lengths of the sides of the triangle are approximately equal to the spins of links of the spin network.

And here is some magic. Remember the constraints on the spins of the 3-valent node? There has to be an intertwiner for this to be a valid spin network state, and it only happens when

1. $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$,
2. $j_1 + j_2 + j_3 \in \mathbb{Z}$.

In the light of $l \sim j$ (length is approximately proportional to spin) we can derive the classical limit of the first requirement:

$$|l_1 - l_2| \leq l_3 \leq (l_1 + l_2) .$$

This is exactly the triangle inequality! This leads us to believe that the two requirements on spins of the 3-valent vertex is the quantum-corrected version of the basic ingredient of the classical geometry — the triangle inequality. This is how classical geometry manifests itself elegantly — as a classical limit of the quantized geometry of chunks of space separated by quantized lengths.

Transition amplitudes

Now as we have a definition of $\mathcal{K}_s$ (we're back to 4d) and an interpretation of quantized geometry at our disposal, we would like to solve the last remaining constraint: the Wheeler-deWitt equation.

Unlike the previous two, this is not an easy task. The Hamiltonian constraint has been rigorously defined as an operator on the space $\mathcal{K}_s$, but hasn't been solved in the most general case.

The quantum space $\mathcal{K}_s$ in itself is of major importance: it is a space of kinematical data, or a space on which the geometric partial observables act as self-adjoint operators. $\mathcal{K}_s$ is perfectly sufficient for describing the states of partial observation and we can forget about the last remaining constraint as long as we aren't evaluating the transition amplitudes.

How do we define these transition amplitudes anyway? Its not like we have an external notion of time, since the time in itself is an emergent concept: it is of quantum nature and its place is inside the state in $\mathcal{K}_s$. We adopt the relational interpretation of Quantum Mechanics as a plausible resolution of the problem of time. Time and space are both encoded in the *boundary state* $\Psi \in \mathcal{K}_s$. We are only interested in the inner product

$$(\Omega|\Psi) ,$$

where Ω is the covariant vacuum, or the empty spin network state.

Note the round brackets instead of the usual angular bras and kets. This is because we make a distinction between the kinematical and physical states:

- Kinematical states are put in angular brackets: $|\Psi\rangle$. They are elements of $\mathcal{K}_s$, they describe the quantum information about partial observables. These states don't exist in the physical world, some of them are self-contradictory. They are used for describing the observer-related information, and the probabilities of observing them are given by the physical inner product of the corresponding physical states.

- Physical states are put in round brackets: $|\Psi) = \hat{P}\,|\Psi\rangle$, where $\hat{P}$ is the projection operator. For any two kinematical states $|\Psi\rangle$ and $|\Phi\rangle$ the only physical prediction of the theory is the *correlation* between the two, given by the matrix element of $\hat{P}$:
$$(\Phi|\Psi) = \left\langle \Phi|\hat{P}^\dagger \hat{P}|\Psi \right\rangle = \left\langle \Phi|\hat{P}|\Psi \right\rangle.$$
It is thus in the matrix elements of $\hat{P}$ where the physics is encoded. This correlation is related to the concept of "probability of observing both at the same time, given that one is plausible" through the usual relation
$$\mathrm{Prob}\,(\Psi,\Phi) = |(\Phi|\Psi)|^2 = \left|\left\langle \Phi|\hat{P}|\Psi \right\rangle\right|^2.$$

The only remaining problem is to define $\hat{P}$. Intuitively, $\hat{P}$ is the projection operator on the subspace of the Hamiltonian constraint. It is formally given by
$$\hat{P} = \int_{-\infty}^{\infty} dt\, e^{it\hat{H}},$$
where $\hat{H}$ is the properly ordered Hamiltonian constraint operator. This gives rise to the spinfoam formalism: the physical inner product of two kinematical states (or of a single boundary state if you wish) is given by a sum over the amplitudes of *spinfoams* — two-complexes with spin networks as boundaries.

Just like superpositions of spin networks define the quantum geometry of spatial slices, the superposition of spinfoams defines the geometry of quantum spacetime.

In Euclidean 3d theory, for example, these are the Ponzano-Regge spinfoams which I have described earlier. In the physical 4d theory there is a version of the spinfoam formalism based on the simplicity constraint and the group-theoretical construction called the Υ_γ map which I hope to cover later.