Slavnov-Tailor identity derivation

Consider a non-abelian gauge quantum field theory defined by its action

$$S[\underline{A}] = \int\limits_x \mathcal{L}(A, \partial A)$$

invariant under gauge transformations of form

$$\underline{A} \to g \circ \underline{A} = g\underline{A}g^{-1} + gdg^{-1},$$

$$\delta_{\text{gauge}}\,\underline{A} = (1+\varepsilon)\circ\underline{A} - \underline{A} = -\left[\underline{A},\,\varepsilon\right] - d\varepsilon = -\nabla\varepsilon.$$

and the functional measure defined by the gauge-invariant norm

$$DA: \qquad \|\underline{\delta A}\|^2 = \operatorname{tr}\int\limits_x \delta A \wedge *\delta A.$$

According to Faddeev & Popov, we could insert in the path integral a delta-functional compensated by the gauge-invariant functional determinant Δ of the Faddeev-Popov operator:

$$\int DA\, f[A] = \int DA\, \delta\left[F^a(x) - \lambda^a(x)\right]\, \Delta[A]\, f[A],$$

where

$$\Delta[A] = \operatorname{Det}\left\{\frac{\delta F^a}{\delta\varepsilon^b}\right\}$$

and arbitrary coordinates $\lambda^a(x)$ parametrize the gauge-fixing hypersurface in the functional space.

Functional coordinate transformations

Consider an infinitesimal gauge transformation $g = 1 + \varepsilon$. We calculate the change $\delta\lambda^a$ which would compensate g in order for the gauge-fixing condition $F^a - \lambda^a = 0$ to hold:

$$F^a[A] - \lambda^a = 0, \qquad F^a[g\circ A] - \lambda^a - \delta\lambda^a = 0,$$

$$F^a[g\circ A] - \lambda^a - \delta\lambda^a = F^a[A] - \lambda^a + \int\limits_y \left\{\frac{\delta F^a(x)}{\delta\varepsilon^b(y)}\right\}\varepsilon^b(y) - \delta\lambda^a(x) = 0,$$

$$\delta\lambda^a(x) = \int\limits_y \left\{\frac{\delta F^a(x)}{\delta\varepsilon^b(y)}\right\}\varepsilon^b(y).$$

The gauge-fixing term

Consider now the functional integral

$$\int DA\, f[A] = \int DA\, \delta\left[F^a(x) - \lambda^a(x)\right]\, \Delta[A]\, f[A].$$

Note that this equation holds for *any* choice of $\lambda^a(x)$, thus we can integrate it over $D\lambda^a$ with *arbitrary* weight, which for convenience we choose to be Gaussian:

$$\int DA\, f[A] = \int DA\, \delta\left[F^a(x) - \lambda^a(x)\right]\, \Delta[A]\, f[A] =$$

$$= \int DA \int D\lambda^a\, \exp\left\{i\int\limits_x\left(-\frac{1}{2}\lambda^a\lambda^a\right)\right\}\delta\left[F^a(x) - \lambda^a(x)\right]\, \Delta[A]\, f[A] =$$

$$= \int DA\,\Delta[A]\,\exp\left\{ i \int\limits_x \left(-\frac{1}{2} F^a F^a \right) \right\} f[A].$$

The gauge-fixing term

$$\mathcal{L}_{\text{gf}}(A, \partial A) = -\frac{1}{2} F^a[A](x) F^a[A](x)$$

appears naturally in the Lagrangian of the theory as a part of the gauge-invariant measure, written in the gauge defined by the form of $F^a[A]$.

Slavnov-Tailor identity

Consider now the generating functional

$$Z[j^{\mu a}] = \int DA\,\Delta[A]\,\exp\left\{ i \int\limits_x \left(\mathcal{L} + \mathcal{L}_{\text{gf}} + j^{\mu a} A^a_\mu \right) \right\} = \left\langle \exp\left\{ i \int\limits_x j^{\mu a} A^a_\mu \right\} \right\rangle$$

and its variation under an inifinitesimal gauge transformation, which we know is equal to zero (because we had a gauge-invariant path integral to begin with):

$$\forall \varepsilon(x): \quad 0 = \delta Z[j^{\mu a}] = i \int\limits_x \left\langle \delta \mathcal{L}_{\text{gf}} + j^{\mu a} \delta A^a_\mu \right\rangle =$$

$$= i \int\limits_{x,y} \left\langle -F^a(x) \left\{ \frac{\delta F^a(x)}{\delta \varepsilon^b(y)} \right\} \right\rangle \varepsilon(y) - i \int\limits_x \left\langle j^{\mu a} \nabla_\mu \varepsilon^a \right\rangle =$$

$$= -i \int\limits_{x,y} \left\langle F^a \delta(x,y) + j^{\mu b} \nabla_\mu \left\{ \frac{\delta F^a(x)}{\delta \varepsilon^b(y)} \right\}^{-1} \right\rangle \delta\lambda^a(y).$$

Thus, we have the identity:

$$\left\langle F^a(x)\delta(x,y) + j^{\mu b} \nabla_\mu \left\{ \frac{\delta F^a(x)}{\delta \varepsilon^b(y)} \right\}^{-1} \right\rangle = 0.$$

Functional variations of this identity with respect to $j^{\mu a}$ taken at $j = 0$ give the coordinate-space Slavnov-Tailor identities.