Self-dual connection

First-order formalism

Consider the first-order formalism of General Relativity. The gravitational field is described by two 1-forms:

- The $\mathbb{R}^4$-valued frame form $\underline{e}^I(x) = e^I_\mu(x)dx^\mu$ describes the metric properties of spacetime. It induces the metric tensor on the tangent space via
$$g_{\mu\nu}(x) = \eta_{IJ}e^I_\mu(x)e^J_\nu(x).$$
- The $SO(3,1)$-valued spin connection form $\underline{\omega}^I{}_J(x) = \omega^I_{\mu J}(x)dx^\mu$ describes the affine properties of spacetime. It provides the $SO(3,1)$-valued covariant derivative which acts as
$$\nabla_\mu = \partial_\mu + \omega_\mu, \quad \nabla f = df + \underline{\omega} \wedge f.$$

The formalism enjoys two kinds of gauge transformations:

- Diffeomorphisms or coordinate transformations $x \to y(x)$ act on 1-forms according to
$$e^I_\mu(x) \to \frac{\partial x^\nu}{\partial y^\mu} \cdot e^I_\nu(x(y)),$$
$$\omega^I_{\mu J}(x) \to \frac{\partial x^\nu}{\partial y^\mu} \cdot \omega^I_{\nu J}(x(y)).$$
- Local (gauge) $SO(3,1)$ rotations $\Omega^I{}_J(x)$ act on fields according to
$$\underline{e}^I(x) \to \Omega^I{}_J(x) \cdot \underline{e}^I(x),$$
$$\underline{\omega}^I{}_J(x) \to \Omega^I{}_K(x) \cdot \underline{\omega}^K{}_L(x) \cdot \Omega^L{}_J(x) + \Omega^I{}_K(x) \cdot \underline{d\Omega}^K{}_J(x).$$
The first equation is simply a rotation of a quantity which lies in the defining representation 4 of $SO(3,1)$, whereas the second equation is nonlinear and is analogous to the gauge connection transformation law.

Palatini action

The Palatini action for the gravitational field is given by

$$S[e,\omega] = \frac{\epsilon_{IJKL}}{16\pi G} \int \underline{e}^I \wedge \underline{e}^J \wedge \left(\underline{\underline{R}}^{KL} + \frac{\Lambda}{2} \cdot \underline{e}^K \wedge \underline{e}^L\right),$$

$$\underline{\underline{R}}^I{}_J = \underline{\underline{d\omega}}^I{}_J + \underline{\omega}^I{}_K \wedge \underline{\omega}^K{}_J,$$
$$R_{\mu\nu}{}^I{}_J = \partial_{[\mu}\omega^I_{\nu]J} + [\omega_\mu;\omega_\nu]^I{}_J.$$

We now variate it with respect to e and ω.

The $\delta\omega$ variation gives the vanishing of torsion $\underline{\underline{T}}^I = \nabla \underline{e}^I$ on the equations of motion:

$$\delta S = \frac{1}{16\pi G}\int_M \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge \left(d\underline{\delta\omega}^{KL} + \underline{\delta\omega}^K{}_P \wedge \underline{\omega}^{PL} + \underline{\omega}^K_P \wedge \underline{\delta\omega}^{PL}\right)$$

$$\underline{\omega}^K_P \wedge \underline{\delta\omega}^{PL} = -\underline{\delta\omega}^{PL} \wedge \underline{\omega}^K_P = \underline{\delta\omega}^{LP} \wedge \underline{\omega}^K_P = -\underline{\delta\omega}^L{}_P \wedge \underline{\omega}^{PK}$$

$$\epsilon_{IJKL}\left(\underline{\delta\omega}^K{}_P \wedge \underline{\omega}^{PL} - \underline{\delta\omega}^L{}_P \wedge \underline{\omega}^{PK}\right) = 2\epsilon_{IJKL} \cdot \underline{\delta\omega}^K{}_P \wedge \underline{\omega}^{PL}$$

$$\delta S = \frac{1}{16\pi G}\int \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge d\underline{\delta\omega}^{KL} + \frac{2}{16\pi G}\int \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge \underline{\delta\omega}^K{}_P \wedge \underline{\omega}^{PL} =$$

$$= \frac{1}{16\pi G}\int \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge d\underline{\delta\omega}^{KP} \cdot \delta^L_P + \frac{2}{16\pi G}\int \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge \underline{\omega}^L{}_P \wedge \underline{\delta\omega}^{KP} =$$

$$= \frac{-2\delta^L_P}{16\pi G}\int \epsilon_{IJKL} d\underline{e}^I \wedge \underline{e}^J \wedge \underline{\delta\omega}^{KP} \cdot \delta^L_P + \frac{2}{16\pi G}\int \epsilon_{IJKL}\underline{e}^I \wedge \underline{e}^J \wedge \underline{\omega}^L{}_P \wedge \underline{\delta\omega}^{KP} =$$

$$= -\frac{\epsilon_{IJKL}}{8\pi G}\int \left(d\underline{e}^I \wedge \underline{e}^J \cdot \delta^L_P + \underline{\omega}^L{}_P \wedge \underline{e}^I \wedge \underline{e}^J\right) \wedge \underline{\delta\omega}^{KP} = 0.$$

$$d\underline{e}^{[I} \cdot \delta^{L]}_P + \underline{\omega}^{[L}{}_P \wedge \underline{e}^{I]} = 0 \qquad \big|\times \delta^I_P$$

$$d\underline{e}^L + \underline{\omega}^L{}_P \wedge \underline{e}^P = \nabla \underline{e}^L = \underline{\underline{T}}^L = 0.$$

The δe variation gives Einstein's equations:

$$\delta S = \frac{1}{16\pi G}\int_M \epsilon_{IJKL}\left(\underline{\delta e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}^{KL} + \underline{e}^I \wedge \underline{\delta e}^J \wedge \underline{\underline{R}}^{KL} + \frac{\Lambda}{2}\cdot \delta\left[\underline{e}^I \wedge \underline{e}^J \wedge \underline{e}^K \wedge \underline{e}^L\right]\right) =$$

$$= \frac{1}{16\pi G}\int_M \epsilon_{IJKL}\left(2\underline{\delta e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}^{KL} + 2\Lambda \cdot \underline{\delta e}^I \wedge \underline{e}^J \wedge \underline{e}^K \wedge \underline{e}^L\right) = 0.$$

$$\epsilon_{IJKL}\left(\underline{e}^J \wedge \underline{\underline{R}}[\omega]^{KL} + \Lambda \cdot \underline{e}^J \wedge \underline{e}^K \wedge \underline{e}^L\right) = 0.$$

Self-dual connection

In complex numbers, the Lie group $SO(3,1) = SO(4)$ is equivalent to two copies of $SO(3)$. These are called the self-dual and anti-self-dual subalgebras.

The self-dual projector $P^i{}_{JK}$is defined via

$$P^i{}_{00} = 0, \quad P^i{}_{0k} = -P^i{}_{k0} = \frac{i}{2}\delta^i_k, \quad P^i{}_{jk} = \frac{1}{2}\epsilon^i{}_{jk}.$$

By applying it to $\underline{\omega}^{IJ}$ we obtain the self-dual complex connection (Ashtekar connection):

$$\underline{A}^i = P^i{}_{JK}\underline{\omega}^{JK}:$$

$$\begin{cases} \underline{A}^1 = \underline{\omega}^{23} + i\underline{\omega}^{01}, \\ \underline{A}^2 = \underline{\omega}^{31} + i\underline{\omega}^{02}, \\ \underline{A}^3 = \underline{\omega}^{12} + i\underline{\omega}^{03}. \end{cases}$$

The anti-self-dual complex connection is equal to $\underline{A}^{*i}$ because $\underline{\omega}^{IJ}$ is real. Thus we have traded 6 real components for 3 complex components in our description of the gravitational connection.

The inverse transformation is given by

$$\begin{cases} \underline{\omega}^{0i} = -\underline{\omega}^{i0} = \mathrm{Im}\underline{A}^i, \\ \underline{\omega}^{jk} = \epsilon^{jk}{}_i \cdot \mathrm{Re}\underline{A}^i. \end{cases}$$

The curvature of the Ashtekar connection is defined to be

$$\underline{\underline{F}}^i = \underline{\underline{dA}}^i + \epsilon^i{}_{jk}\cdot \underline{A}^j \wedge \underline{A}^k.$$

In fact, we claim that the curvature of self-dual connection is equal to the self-dual part of the curvature of the real connection. Indeed:

$$P^i{}_{JK}\underline{\underline{R}}^{JK} = P^i{}_{JK}\left(\underline{\underline{d\omega}}^{JK} + \underline{\omega}^J{}_L \wedge \underline{\omega}^{LK}\right) = \underline{\underline{dA}}^i + P^i{}_{JK}\underline{\omega}^J{}_L \wedge \underline{\omega}^{LK} =$$

$$= \underline{\underline{dA}}^i + \frac{i}{2}\delta^i_k\underline{\omega}^{0l} \wedge \underline{\omega}^{lk} - \frac{i}{2}\delta^i_j\underline{\omega}^{jl} \wedge \underline{\omega}^{l0} + \frac{1}{2}\epsilon^i{}_{jk}\underline{\omega}^{jl} \wedge \underline{\omega}^{lk} - \frac{1}{2}\epsilon^i{}_{jk}\underline{\omega}^{j0} \wedge \underline{\omega}^{0k} =$$

$$= \underline{\underline{dA}}^i + \frac{i}{2}\epsilon^{ikl}\mathrm{Im}\underline{A}^l \wedge \mathrm{Re}\underline{A}^k + \frac{i}{2}\epsilon^{ilk}\mathrm{Re}\underline{A}^k \wedge \mathrm{Im}\underline{A}^l + \frac{1}{2}\epsilon^{jki}\epsilon^{jla}\epsilon^{lkb}\mathrm{Re}\underline{A}^a \wedge \mathrm{Re}\underline{A}^b + \frac{1}{2}\epsilon^{ijk}\mathrm{Im}\underline{A}^j \wedge \mathrm{Im}\underline{A}^k =$$

$$= \underline{\underline{dA}}^i + \frac{i}{2}\epsilon^{ikl}\mathrm{Im}\underline{A}^l \wedge \mathrm{Re}\underline{A}^k + \frac{i}{2}\epsilon^{ilk}\mathrm{Re}\underline{A}^k \wedge \mathrm{Im}\underline{A}^l + \frac{1}{2}\left(\delta^{kl}\delta^{ia} - \delta^{ka}\delta^{il}\right)\epsilon^{lkb}\mathrm{Re}\underline{A}^a \wedge \mathrm{Re}\underline{A}^b + \frac{1}{2}\epsilon^{ijk}\mathrm{Im}\underline{A}^j \wedge \mathrm{Im}\underline{A}^k =$$

$$= \underline{\underline{dA}}^i + \frac{i}{2}\epsilon^{ilk}\mathrm{Im}\underline{A}^k \wedge \mathrm{Re}\underline{A}^l + \frac{i}{2}\epsilon^{ilk}\mathrm{Re}\underline{A}^k \wedge \mathrm{Im}\underline{A}^l + \frac{1}{2}\epsilon^{ilk}\mathrm{Re}\underline{A}^k \wedge \mathrm{Re}\underline{A}^l - \frac{1}{2}\epsilon^{ilk}\mathrm{Im}\underline{A}^k \wedge \mathrm{Im}\underline{A}^l =$$

$$= \underline{\underline{dA}}^i + \frac{i}{2}\epsilon^{ikl}\left(\mathrm{Re}\underline{A}^k + i\mathrm{Im}\underline{A}^k\right) \wedge \left(\mathrm{Re}\underline{A}^l + i\mathrm{Im}\underline{A}^l\right) = \underline{\underline{dA}}^i + \frac{i}{2}\epsilon^{ijk}\underline{A}^j \wedge \underline{A}^k = \underline{\underline{F}}^i.$$

Self-dual dynamics

The term in the Lagrangian, which gives the dynamics of the gravitational field

$$\epsilon_{IJKL} \cdot \underline{e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}^{KL}$$

has to be rewritten in terms of the self-dual connection. The answer is

$$iP_{iIJ}\underline{e}^I \wedge \underline{e}^J \wedge F^i = iP_{iIJ}P^i{}_{KL}\underline{e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}^{KL} = \epsilon_{IJKL} \cdot \underline{e}^I \wedge \underline{e}^J \wedge \underline{\underline{R}}^{KL}.$$

In order to prove this, we need to proove the following identity:

$$P_{iIJ}P^i{}_{KL} \cdot F^{[IJKL]} = -i\epsilon_{IJKL} \cdot F^{[IJKL]}$$

Consider all four special cases:

1. Both I, J are zero or both K, L are zero. This renders zeroes on both parts and the equation holds.
2. One of I, J is zero and the other one is not, same for K, L. Without loss of generality let $I = 0$ and $K = 0$:

$$P_{iIJ}P^i{}_{KL} \cdot F^{[IJKL]} = -\frac{1}{4}\delta^i_J\delta^i_L \cdot F^{[IJKL]} = -\frac{1}{4}\delta_{JL} \cdot F^{[IJKL]} = 0.$$

3. One of I, J is zero and both K and L are nonzero:

$$P_{iIJ}P^i{}_{KL} \cdot F^{[IJKL]} = \frac{i}{4}\epsilon^{JKL} \cdot F^{[0JKL]} = i\epsilon^{IJKL}F^{[IJKL]} = -i\varepsilon_{IJKL}F^{[IJKL]}.$$

4. All indices I, J, K, L are nonzero:

$$P_{iIJ}P^i{}_{KL} \cdot F^{[IJKL]} = \frac{1}{4}\epsilon^{iIJ}\epsilon^{iKL} \cdot F^{[IJKL]} = \frac{1}{4}\left(\delta^{IK}\delta^{JL} - \delta^{IL}\delta^{JK}\right) \cdot F^{[IJKL]} = 0.$$

Thus the equation is proven and the dynamics of the self-dual connection is described by the self-dual action:

$$S[e, A] = \frac{1}{16\pi G}\int \left(iP_{iIJ}\underline{e}^I \wedge \underline{e}^J \wedge F^i + \frac{\Lambda}{2}\epsilon_{IJKL} \cdot \underline{e}^I \wedge \underline{e}^J \wedge \underline{e}^K \wedge \underline{e}^L\right).$$