Path integral for point particle

Worldline diffeomorphisms

The worldline of the point particle is a one-dimensional manifold with the topology of a line. Therefore we can parametrize the worldline by a single coordinate $\tau \in \mathbb{R}$. A transformation of coordinates is an element of the *diffeomorphism group* and can be represented by an invertible function

$$\tau \to f(\tau). \tag{1}$$

The group structure of diffeomorphisms is given by the following relations:

1. The group product is represented by the composition of functions:

$$h = g \circ f \implies h(\tau) = g(f(\tau)); \tag{2}$$

2. The identity diffeomorphism does not change the coordinate:

$$e : \; e(\tau) = \tau; \tag{3}$$

3. The inverse diffeomorphism is given by the inverse function:

$$f = g^{-1} \implies f \circ g = e \implies f(g(\tau)) = \tau. \tag{4}$$

Infinitesimal diffeomorphisms are given by

$$f_\varepsilon(\tau) = e + \varepsilon \cdot \xi(\tau) + o(\varepsilon), \tag{5}$$

where ε parameterizes the 1-dimensional subgroup which goes through the group identity at $\varepsilon = 0$, $f(\varepsilon)$ is an element of the diffeomorphism group in the defining representation, ξ is an element of the Lie algebra of the diffeomorphism group.

According to the principle of general covariance, all the objects in the theory lie in some representations of the diffeomorphism group. The most commonly used representations include:

- *Scalar fields* — functions $\varphi(\tau)$ with an arbitrary diffeomorphism $f(\tau)$ acting on them according to

$$\varphi(\tau) \to f(\tau) \circ \varphi(\tau) = \varphi(f(\tau)).$$

  E.g. the coordinates $x^\mu(\tau)$ which parameterize the embedding of the worldline in spacetime transform under worldline diffeomorphisms as scalar fields.

- *Covariant fields* — functions $\zeta(\tau)$ with an arbitrary diffeomorphism $f(\tau)$ acting on them according to

$$\zeta(\tau) \to f(\tau) \circ \zeta(\tau) = \frac{df}{d\tau} \cdot \zeta(f(\tau));$$

- *Contravariant fields* — functions $\omega(\tau)$ with an arbitrary diffeomorphism $f(\tau)$ acting on them according to

$$\omega(\tau) \to f(\tau) \circ \omega(\tau) = \left(\frac{df}{d\tau}\right)^{-1} \cdot \omega(f(\tau)).$$

- *Fields of weight w* — a more general case, functions $q_w(\tau)$ with an arbitrary diffeomorphism $f(\tau)$ acting on them according to

$$q_w(\tau) \to f(\tau) \circ q_w(\tau) = \left(\frac{df}{d\tau}\right)^{w} \cdot q_w(f(\tau)).$$

  Scalar, covariant and contravariant fields are special cases of fields with weights 0, 1 and -1 accordingly.

The Lie algebra element $\xi(\tau)$ corresponds to an infinitesimal coordinate variation $\delta\tau/\varepsilon$, and is thus a contravariant field. Thus, we can compute the adjoint action of the diffeomorphism Lie algebra:

$$f(\tau) \circ \chi(\tau) = (\tau + \varepsilon \cdot \xi(\tau) + o(\varepsilon)) \circ \chi(\tau) = \left(\frac{d}{d\tau}\left[\tau + \varepsilon \cdot \xi(\tau) + o(\varepsilon)\right]\right)^{-1} \cdot \chi(\tau + \varepsilon \cdot \xi(\tau) + o(\varepsilon)) =$$

$$= \left(1 + \varepsilon \cdot \xi'(\tau) + o(\varepsilon)\right)^{-1} \cdot \left(\chi(\tau) + \varepsilon \cdot \chi'(\tau)\xi(\tau) + o(\varepsilon)\right) =$$
$$= \left(1 - \varepsilon \cdot \xi'(\tau) + o(\varepsilon)\right) \cdot \left(\chi(\tau) + \varepsilon \cdot \chi'(\tau)\xi(\tau) + o(\varepsilon)\right) =$$
$$= \chi(\tau) + \varepsilon \cdot \left(\xi(\tau)\,\chi'(\tau) - \chi(\tau)\,\xi'(\tau)\right) + o(\varepsilon).$$

Where we have used the trivial relation

$$\left(a + b\varepsilon + o(\epsilon)\right)^{-1} = a - b\varepsilon + o(\epsilon),$$

which can be checked by simple substitution.

Thus we have the following Lie bracket:

$$[\xi(\tau),\, \chi(\tau)] = \xi(\tau)\,\chi'(\tau) - \chi(\tau)\,\xi'(\tau).$$

Note that this is a special case of a more general Lie algebra of the diffeomorphism group for an n-dimensional manifold. The Lie bracket is given by a Lie derivative of vector fields.

We would also need a notion of the tangent space taken at an arbitrary group element $f \neq e$, which in the defining representations consists of small variations $\delta f(\tau)$. We find the following relation to the corresponding Lie algebra element:

$$f + \delta f = f \circ (e + \varepsilon \cdot \xi) \implies e + f^{-1} \circ \delta f = e + \varepsilon \cdot \xi \implies \varepsilon \cdot \xi = f^{-1} \circ \delta f. \tag{6}$$

Relativistic particle

The dynamics of the relativistic particle is given by the action functional

$$S[x^\mu(\tau)] = -m \int\limits_0^1 \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\tau. \tag{7}$$

The first variation of this functional gives the classical equation of motion:

$$\delta S = -m\delta \int\limits_0^1 \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\tau = -m \int\limits_0^1 d\tau\, \frac{\eta_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} =$$
$$= \left.\frac{-m\eta_{\mu\nu}\dot{x}^\mu \delta x^\nu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\right|_{\tau=0}^{\tau=1} + m \int\limits_0^1 d\tau\, \frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{x}^\mu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\right)\delta x^\nu,$$
$$\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{x}^\mu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\right) \simeq 0. \tag{8}$$

This equation is quite complex and nonlinear. It enjoys a vast number of physically indistinguishable solutions related by diffeomorphisms. We could fix the gauge by demanding that τ actually measures the proper time along the world line:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = d\tau^2 \implies \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1,$$
$$\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{x}^\mu}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\right) = \eta_{\mu\nu}\ddot{x}^\mu = 0 \implies \ddot{x}^\mu = 0.$$

Thus the equation of motion gives straight worldlines as solutions.

The original action (7) contains the square root and thus isn't suited for quantum physics. We introduce instead another action:

$$S[x^\mu(\tau), e(\tau)] = -\frac{1}{2}\int\limits_0^1 d\tau \left(\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{e} + em^2\right), \tag{9}$$

where $x^\mu(\tau)$ defines the embedding of the worldline in spacetime and $e(\tau)$ is an auxillary covariant field. We can check that when $e(\tau)$ is covariant, the action ,$S[x, e]$ is invariant under the action of diffeomorphisms.

Both actions (7) and (9) give the same physics. Consider a variation in δe:

$$\delta S = \frac{1}{2} \int_0^1 d\tau \left(\frac{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - m^2 \right) \delta e(\tau) = 0 \implies e(\tau) = \frac{1}{m} \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu};$$

$$S = -\frac{1}{2} \int_0^1 d\tau \left(\frac{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} + e m^2 \right) = -\frac{1}{2} \int_0^1 d\tau \left(\frac{m \, \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} + m \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right) = -m \int_0^1 d\tau \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}.$$

Thus, after the substitution of an algebraic equation of motion for $e(\tau)$ the second action turns into the first. As shown by Polyakov, since the equation of motion for e is algebraic (not differential), both actions give the same quantum physics.

Functional measure

The functional measure is defined as a regularized Riemannian measure generated by the following scalar products:

$$\|\delta x\|^2 = \eta_{\mu\nu} \int_0^1 d\tau \, e(\tau) \, \delta x^\mu(\tau) \, \delta x^\nu(\tau); \tag{10}$$

$$\|\delta e\|^2 = \int_0^1 d\tau \, e^{-1}(\tau) \, \delta e^2(\tau). \tag{11}$$

The powers of e are such that both products are diffeomorphism-invariant. In the following we explicitly extract the integration over the functional space of diffemorphisms in order to obtain a functional measure on the space of gauge orbits.

First, we will have to define a functional measure on the space of diffeomorphisms. Consider an arbitrary diffeomorphism f and its small variation δf. According to (6), the corresponding Lie algebra element ξ is equal to

$$\xi = f^{-1} \circ \delta f \implies \xi(\tau) = \delta f(f^{-1}(\tau)) \implies \delta f(\tau) = \xi(f(\tau)),$$

and transforms as a contravariant field. Thus, a suitable diffeomorphism-invariant scalar product on the functional space of diffeomorphisms is

$$\|\delta f\|^2 = \int_0^1 d\tau \, e^3(\tau) \, \xi^2(\tau). \tag{12}$$

The only invariant of the 1-dimensional Riemannian geometry of the worldline is the proper time

$$L[e] = \int_0^1 d\tau \, \sqrt{g_{\tau\tau}(\tau)} = \int_0^1 d\tau e(\tau). \tag{13}$$

Thus, it is always possible to pass to a particular choice of parametrization — the *homogeneous coordinate* σ defined by

$$e(\sigma) = \text{const} = L.$$

Coordinates τ and σ are related by the covariant transformation property of e:

$$e(\tau) = \frac{df}{d\tau} e(\sigma) = \frac{df}{d\tau} L.$$

The variation of e in the coordinate τ is thus equal to

$$\delta e(\tau) = \frac{d\delta f}{d\tau} L + \frac{df}{d\tau} \delta L.$$

We substitute this in (11):

$$\|\delta e\|^2 = \int_0^1 d\tau \, e^{-1}(\tau) \, \delta e^2(\tau) = \int_0^1 d\tau \, \left(\frac{df}{d\tau} \right)^{-1} L^{-1} \left(\frac{d\delta f}{d\tau} L + \frac{df}{d\tau} \delta L \right)^2 =$$

$$= L^{-1} \int_0^1 d\tau \left(\frac{df}{d\tau}\right)^{-1} \left(\frac{df}{d\tau}\right)^2 \left(\left(\frac{df}{d\tau}\right)^{-1} \frac{d\delta f}{d\tau} L + \delta L \right)^2 =$$

$$= L^{-1} \int_0^1 d\sigma \left(\frac{d\tau}{d\sigma} \frac{d\delta f(\tau)}{d\tau} L + \delta L \right)^2 = L \int_0^1 d\sigma \left(\frac{d\delta f(\tau)}{d\sigma} + \frac{\delta L}{L} \right)^2 =$$

$$= L \int_0^1 d\sigma \left(\frac{d\delta f(\tau)}{d\sigma} \right)^2 + 2\delta L \int_0^1 d\sigma \frac{d\delta f(\tau)}{d\sigma} + \frac{\delta L^2}{L} \int_0^1 d\sigma = L \int_0^1 d\sigma \left(\frac{d\xi(\sigma)}{d\sigma} \right)^2 + 2\delta L \int_0^1 d\sigma \frac{d\xi(\sigma)}{d\sigma} + \frac{\delta L^2}{L} =$$

$$= -L \int_0^1 d\sigma \xi(\sigma) \frac{d^2}{d\sigma^2} \xi(\sigma) + 0 + \frac{\delta L^2}{L} = \int_0^1 d\sigma \xi(\sigma) \left[-L \frac{d^2}{d\sigma^2} \right] \xi(\sigma) + \frac{\delta L^2}{L}.$$

Now we have all the necessary ingredients to define De — a part of the functional measure responsible for the integration over e:

$$De = \sqrt{L^{-1} \det \left[-L \frac{d^2}{d\sigma^2} \right]} \cdot \prod_\tau d\xi(\tau) \cdot dL.$$

For diffeomorphisms, we have:

$$\|\delta f\|^2 = \int_0^1 d\tau\, e^3(\tau)\, \xi^2(\tau) = \int_0^1 d\tau \left(\frac{df}{d\tau} \right)^3 L^3\, \xi^2(\tau) = L^3 \int_0^1 d\sigma \left(\frac{d\sigma}{d\tau} \xi(\tau) \right)^2 = L^3 \int_0^1 d\sigma \xi(\sigma)^2;$$

$$Df = \sqrt{\det [L^3]} \cdot \prod_\tau d\xi(\tau).$$

Thus, the functional measure on the quotient space of gauge orbits is

$$\frac{De}{Df} = \sqrt{\frac{\det \left[-L \frac{d^2}{d\sigma^2} \right]}{L \cdot \det [L^3]}} \cdot dL = \sqrt{L^{-1} \det \left[-\frac{1}{L^2} \frac{d^2}{d\sigma^2} \right]} \cdot dL. \tag{14}$$

Regularization

Now we have to make sense of the results obtained in the previous section by regularizing functional determinants of form

$$\det A = \prod_n \lambda_n = \exp \left\{ \sum_n \log \lambda_n \right\}.$$

We choose to regularize by the *generalized Riemann's zeta function*:

$$\zeta_A(s) = \sum_n \lambda_n^{-s}.$$

Since

$$\frac{d}{ds} \lambda_n^{-s} \bigg|_{s=0} = -\lambda_n^{-s} \log \lambda_n \big|_{s=0} = -\log \lambda_n,$$

we have

$$\det A = \exp \left\{ \sum_n \log \lambda_n \right\} \to e^{-\zeta_A'(0)}.$$

Our functional measure contains the following determinant:

$$\det A = \det \left[-\frac{1}{L^2} \frac{d^2}{d\sigma^2} \right].$$

First, we have to find a collection of eigenvalues for A:

$$A\psi = \lambda\psi \implies -\frac{1}{L^2} \frac{d^2 \psi(\sigma)}{d\sigma^2} = \lambda \psi(\sigma) \tag{15}$$

with the Dirichlet boundary conditions (since $\tau_A = 0,\ \tau_B = 1$):

$$\psi(0) = 0,\ \psi(1) = 0.$$

The general solution is given by

$$\psi(\sigma) = a \sin\left[L\sqrt{\lambda}\sigma\right] + b\cos\left[L\sqrt{\lambda}\sigma\right],$$

which satisfies the boundary confitions when $b = 0$ and $L\sqrt{\lambda} = \pi n$ where $n \in \mathbb{N}$. We obtain the collection of eigenvalues:

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2.$$

The generalized Riemann's zeta function renders

$$\zeta_A(s) = \sum_n \lambda_n^{-s} = \sum_n \left(\frac{\pi n}{L}\right)^{-2s} = \left(\frac{L}{\pi}\right)^{2s} \cdot \zeta(2s),$$

where $\zeta(2s)$ is the ordinary Riemann's zeta function. The regularized determinant renders

$$\zeta_A'(0) = \log\left[\frac{1}{2L}\right] = -\log[2L];$$

$$\det\left[-\frac{1}{L^2}\frac{d^2}{d\sigma^2}\right] = e^{-\zeta_A'(0)} = e^{\log[2L]} = 2L.$$

Thus, the measure on the space of gauge orbits is equal to

$$\frac{De}{Df} = \sqrt{L^{-1}\det\left[-\frac{1}{L^2}\frac{d^2}{d\sigma^2}\right]} \cdot dL = \sqrt{L^{-1}\cdot 2L}\cdot dL = \sqrt{2}dL \sim dL.$$

The constant multiplier $\sqrt{2}$ gets absorbed into the redefinition of the functional measure, the final result being

$$\frac{De}{Df} = dL. \tag{16}$$

Computing the path integral

The propagator is given by

$$W_{AB} = \int \frac{De}{Df}\int Dx\, e^{S[x,e]} = \int \frac{De}{Df}\int Dx\, \exp\left\{-\frac{1}{2}\int_0^1 d\tau\left(\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{e(\tau)} + m^2 e(\tau)\right)\right\} = \int_0^\infty dL Q_{AB}(L).$$

In this section we compute the diffeomorphism-invariant Gaussian Dx path interal

$$Q_{AB}(L) = \int Dx\, \exp\left\{-\frac{1}{2}\int_0^1 d\tau\left(\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{e(\tau)} + m^2 e(\tau)\right)\right\}.$$

Because of diffeomorphism invariance we are free to choose any coordinate system, the most convenient being the homogeneous coordinate σ:

$$e(\sigma) = \text{const} = L;$$

$$\|\delta x\|^2 = \eta_{\mu\nu}\int_0^1 d\tau\, e(\tau)\,\delta x^\mu(\tau)\,\delta x^\nu(\tau) = L\eta_{\mu\nu}\int_0^1 d\sigma\,\delta x^\mu \delta x^\nu;$$

$$Dx = \prod_\sigma L^{d/2} d^d x(\sigma);$$

$$S[x,e] = -\frac{1}{2}\int_0^1 d\sigma\left(\frac{\eta_{\mu\nu}}{L}\dot{x}^\mu\dot{x}^\nu + m^2 L\right) = -\frac{\eta_{\mu\nu}}{2L}\int_0^1 d\sigma \dot{x}^\mu\dot{x}^\nu - \frac{m^2 L}{2};$$

$$Q_{AB}(L) = \int Dx \, \exp\{-S[x,e]\} = \int \prod_\sigma L^{d/2} d^d x(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L} \int_0^1 d\sigma \dot{x}^\mu \dot{x}^\nu - \frac{m^2 L}{2} \right\} =$$

$$= e^{-m^2 L/2} \cdot \int \prod_\sigma L^{d/2} d^d x(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L} \int_0^1 d\sigma \dot{x}^\mu \dot{x}^\nu \right\}.$$

We have to compute the Gaussian path integral over $\prod_\sigma d^d x(\sigma)$. We adopt a generalization of the discrete formula:

$$\int d^n x \, \exp\left\{ -\frac{1}{2} A_{ab} x_a x_b + b_a x_a \right\} = \sqrt{\frac{(2\pi)^n}{\det A}} \cdot \exp\left\{ \frac{1}{2} \left[A^{-1}\right]_{ab} b_a b_b \right\}. \tag{17}$$

The boundary conditions $x(0) = A$, $x(1) = B$ can be taken into account by a reparametrization

$$x^\mu(\sigma) = c^\mu(\sigma) + r^\mu(\sigma),$$

where $c^\mu(\sigma)$ is the solution of the corresponding classical equation of motion:

$$c^\mu(\sigma) = A^\mu + (B^\mu - A^\mu)\sigma,$$

and $r(\sigma)$ obeys trivial boundary conditions $r(0) = r(1) = 0$. Also, the functional measure does not change under a shift by $c(\sigma)$ in the functional space:

$$Dx = Dr.$$

Thus

$$\eta_{\mu\nu} \int_0^1 d\sigma \dot{x}^\mu \dot{x}^\nu = \eta_{\mu\nu} \int_0^1 d\sigma \dot{c}^\mu \dot{c}^\nu + 2\eta_{\mu\nu} \int_0^1 d\sigma \dot{c}^\mu \dot{r}^\mu + \eta_{\mu\nu} \int_0^1 d\sigma \dot{r}^\mu \dot{r}^\nu;$$

$$\eta_{\mu\nu} \int_0^1 d\sigma \dot{c}^\mu \dot{c}^\nu = \int_0^1 d\sigma \left(B^\mu - A^\mu\right)\left(B_\mu - A_\mu\right) = \left(B - A\right)^2;$$

$$2\eta_{\mu\nu} \int_0^1 d\sigma \dot{c}^\mu \dot{r}^\mu = 2\eta_{\mu\nu} \dot{c}^\mu r^\nu \big|_0^1 - 2\eta_{\mu\nu} \int_0^1 d\sigma \ddot{c}^\mu r^\nu = 0 - 0 = 0;$$

$$\eta_{\mu\nu} \int_0^1 d\sigma \dot{x}^\mu \dot{x}^\nu = \left(B - A\right)^2 + \eta_{\mu\nu} \int_0^1 d\sigma \dot{r}^\mu \dot{r}^\nu;$$

$$Q_{AB}(L) = e^{-m^2 L/2} \cdot \int \prod_\sigma L^{d/2} d^d x(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L} \int_0^1 d\sigma \dot{x}^\mu \dot{x}^\nu \right\} =$$

$$= \exp\left\{ -\frac{m^2 L}{2} - \frac{(B-A)^2}{2L} \right\} \cdot \int \prod_\sigma L^{d/2} d^d r(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L} \int_0^1 d\sigma \dot{r}^\mu \dot{r}^\nu \right\}.$$

The Dr integral is gaussian and can be taken:

$$\prod_\sigma L^{d/2} \int d^d r(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L} \int_0^1 d\sigma \dot{r}^\mu \dot{r}^\nu \right\} = \prod_\sigma \int d^d q(\sigma) \, \exp\left\{ -\frac{\eta_{\mu\nu}}{2L^2} \int_0^1 d\sigma \dot{q}^\mu \dot{q}^\nu \right\} \sim$$

$$\sim \det\left[-\frac{\eta_{\mu\nu}}{L^2} \frac{d^2}{d\sigma^2} \right]^{-1/2} = \det\left[-\frac{1}{L^2} \frac{d^2}{d\sigma^2} \right]^{-d/2}.$$

The propagator

We have to regularize the results from the last section and do the dL integral.

$$-\frac{1}{L^2}\frac{d^2\psi(\sigma)}{d\sigma^2}=\lambda\psi(\sigma);\;\;\psi(0)=\psi(1)=0.$$

$$\psi(\sigma)=a\sin\left[\sqrt{\lambda}L\sigma\right];$$

$$\sqrt{\lambda}L=\pi n;$$

$$\lambda_n=\frac{\pi^2n^2}{L^2}.$$

$$\zeta_A(s)=\sum\left(\frac{\pi^2n^2}{L^2}\right)^{-s}=\left(\frac{L}{\pi}\right)^{2s}\zeta(2s);$$

$$\zeta_A'(0)=-\log 2-\log L=-\log\left[2L\right];$$

$$\det\left[-\frac{\eta_{\mu\nu}}{L}\frac{d^2}{d\sigma^2}\right]=e^{-\zeta_A'(0)}=2L;$$

$$\det\left[-\frac{1}{L}\frac{d^2}{d\sigma^2}\right]^{-d/2}=(2L)^{-d/2}\sim L^{-d/2}.$$

The dL integral gives

$$W_{AB}=\int_0^\infty dL\,L^{-d/2}\exp\left\{-\frac{m^2L}{2}-\frac{(B-A)^2}{2L}\right\}=\frac{1}{(2\pi)^{d/2}}\left(\frac{\Delta s}{m}\right)^{1-\frac{d}{2}}\cdot K_{\frac{d}{2}-1}\left[m\cdot\Delta s\right],$$

where

$$\Delta s=\sqrt{(B-A)^2}=\sqrt{\eta_{\mu\nu}(B^\mu-A^\mu)(B^\nu-A^\nu)},$$

and $K_n(x)$ is the modified Bessel function.

This is the propagator for the free massive relativistic particle. Its fourier image is the standard Klein-Gordon propagator:

$$P(k)=\int d^dx\,e^{-ik_\mu(B-A)^\mu}\cdot\int_0^\infty dL\,L^{-d/2}\exp\left\{-\frac{m^2L}{2}-\frac{(B-A)^2}{2L}\right\}=\int_0^\infty dL\,L^{-d/2}e^{-m^2L/2}\cdot(2\pi L)^{d/2}\cdot e^{-k^2L/2}=$$

$$=(2\pi)^{d/2}\int_0^\infty dL\,\exp\left\{-\frac{1}{2}L\left(k^2+m^2\right)\right\}\sim\frac{1}{k^2+m^2}.$$