N/A gauge functional measure invariance

We demonstrate that the functional naive functional measure generated by the norm

$$\|\delta A\|^2 = \int dx\, g^{\mu\nu}(x)\delta_{ab}\, \delta A^a_\mu \delta A^b_\nu .$$

is invariant under the infinitesimal gauge transformations of form

$$A^a_\mu \to A^a_\mu + [\nabla_\mu \varepsilon]^a = A^a_\mu + \partial_\mu \varepsilon^a + f^a_{bc}\varepsilon^b A^c_\mu .$$

The generalized "index" which labels the coordinates on the functional space consists of a spacetime point, a covariant spacetime index and a color index:

$$A = \{x \in M,\ \mu, a\}\,; \quad B = \{y \in M,\ \nu, b\}\,; \quad C = \{z \in M,\ \sigma, c\}\,.$$

We thus write down the Riemannian functional metric which generates the mentioned above scalar product as

$$\gamma_{AB} = \delta(x,y)\cdot g^{\mu\nu}\cdot \delta_{ab}.$$

Now consider an infinitesimal gauge transformation parametrized by $\varepsilon^c(x)$. To such a transformation corresponds a vector field in the functional space:

$$V^C = \partial_\sigma \varepsilon^c(z) + f^c_{de}\varepsilon^d(z) A^e_\sigma(z).$$

Now consider a variation of the functional metric γ_{AB} under an infinitesimal gauge transformation. It is given by the Lie derivative in the functional space:

$$\delta_V \gamma_{AB} = V^C \partial_C \gamma_{AB} + \partial_A V^C \cdot \gamma_{CB} + \partial_B V^C \cdot \gamma_{AC}.$$

We expand the functional indices and arrive at the following formula:

$$\delta_V \gamma\,(x\mu a;\, y\nu b) = \int dz \left\{ V^c_\sigma(z)\cdot \frac{\delta\gamma\,(x\mu a;\, y\nu b)}{\delta A^c_\sigma(z)} + \gamma\,(z\sigma c;\, y\nu b)\cdot \frac{\delta V^c_\sigma(z)}{\delta A^a_\mu(x)} + \gamma\,(x\mu a;\, z\sigma c)\cdot \frac{\delta V^c_\sigma(z)}{\delta A^b_\nu(y)} \right\} =$$

$$= \int dz\, \left\{\delta(z,y) g^{\sigma\nu}\delta_{cb}\cdot f^c_{de}\varepsilon^d(z)\cdot \delta^e_a \delta^\mu_\sigma \delta(z,x) + \delta(x,z) g^{\mu\sigma}\delta_{ac}\cdot f^c_{de}\varepsilon^d(z)\cdot \delta^e_b \delta^\nu_\sigma \delta(z,y)\right\} =$$

$$= f^c_{de}\varepsilon^d(x)\delta(x,y) g^{\mu\nu}\cdot\left(\delta_{cb}\delta^e_a + \delta_{ac}\delta^e_b\right) = \varepsilon^d(x)\delta(x,y) g^{\mu\nu}\cdot \left(f^b_{da} + f^a_{db}\right).$$

For compact gauge groups, $f^c_{ab} = f^{cab}$ is totally antisymmetric, thus $f^{bda} + f^{adb} = 0$ and

$$\delta_V \gamma_{AB} = 0$$

right away.

For noncompact groups, gauge transformations are not isometries of the functional space, but they still preserve the functional measure since

$$\mathrm{tr}\delta_V \gamma_{AB} = \delta_V \gamma_{AA} = \delta(0)\cdot \int dx\, g^{\mu\mu}\varepsilon^d \left(f^a_{ad} + f^a_{da}\right) = 0$$

and

$$\delta_V DA \sim \delta_V \det\gamma \sim \mathrm{tr}\delta_V \gamma_{AB} = 0.$$

I guess, this kind of justifies the exploitation of the measure on the functional space in the Faddeev-Popov method, which gives the measure on the space of gauge orbits and the correct Feynman rules for the perturbation theory.