Representations of Lie algebras

I introduce the basics of the generic representation theory for simple and semisimple Lie algebras. The concepts of irreducible representations (irreps), intertwiners, Casimir operators, Cartan subalgebra, root vectors, weight vectors, Dynkin diagrams etc are explained. I mention the general recipy for classifying the finite-dimensional irreducible representations of simple and semisimple Lie algebras, as well as the classification of finite-dimensional simple and semisimple Lie algebras.

Basic definitions

The Lie algebra

A *Lie algebra* $\mathfrak{g}$ over the number field $\mathbb{K}$ is a linear vector space equipped with an antisymmetric bilinear operation satisfying the Jacobi identity called the *Lie bracket:*

1. bilinearity: $\forall a, b \in \mathfrak{g},\ p, q \in \mathbb{K}: \quad [pa + qb, c] = p[a, c] + q[b, c]$,
2. antisymmetricity: $\forall a, b \in \mathfrak{g}: \quad [a, b] = -[b, a]$,
3. Jacobi identity: $\forall a, b, c \in \mathfrak{g}: \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Note that the Lie bracket is not only noncommutative, but also nonassociative. It is the reason why it is denoted $[a, b]$ instead of $a \cdot b$. In what follows we will work with the number field of complexes $\mathbb{C}$ unless explicitly stated otherwise.

Structure constants

Consider a basis in the Lie algebra:

$$\mathfrak{g} \ni a = a^\alpha t_\alpha,$$

where $\alpha \in 1..n$ where $n = \dim \mathfrak{g}$. The basis vectors t_α are called the generators of the algebra. The Lie bracket of any two algebra elements can be expanded over the basis of generators using its bilinearity:

$$[a, b] = [a^\alpha t_\alpha, b^\beta t_\beta] = a^\alpha b^\beta [t_\alpha, t_\beta].$$

Because $\{t_\alpha\}$ is a basis in the Lie algebra, any element of the algebra can be expanded over this basis, the $[t_\alpha, t_\beta]$ being no exception. We arrive at

$$[a, b] = a^\alpha b^\beta f^\gamma_{\alpha\beta} t_\gamma,$$

where the coefficients $f^\gamma_{\alpha\beta}$ are called the *structure constants*: they uniquely determine the structure of the Lie algebra. More precisely, the structure of the algebra is determined by equivalence classes of the structure constants with respect to the changes of the basis. The transformation properties of the matrix $f^\gamma_{\alpha\beta}$ are nicely encoded in the tensor notation: the up and down indices transform under the usual transformation law.

The structure constatnts are antisymmetric with respect to the two down indices. It is of course true in any basis of generators as it is a tensor equation which can hold either in all coordinates at once or in none. The Jacobi identity also imposes an additional constraint on the structure constants.

Representations

An r-dimensional representation ρ of the Lie algebra is a linear function from the algebra to the space of $\mathbb{K}$-valued $r \times r$ matrices:

1. linearity: $\forall a \in \mathfrak{g}, k \in \mathbb{K}: \quad \rho(ka) = k\rho(a)$,
2. algebra structure: $\forall a, b \in \mathfrak{g}: \quad \rho([a, b]) = \rho(a)\rho(b) - \rho(b)\rho(a)$.

As we see, the representation models the Lie bracket as a commutator of linear operators (matrices) on some r-dimensional linear space. The number r is called the dimension of the representation, and in general it is not equal to the dimension of the algebra n!

Of course, a Lie algebra admits multiple representations of different dimensions.

A subspace S of the representation space is called an *invariant subspace* if

$$\forall a \in \mathfrak{g}: \quad \rho(a) S \subseteq S.$$

A representation is called *irreducible* if it doesn't have invariant subspaces except for $\{0\}$ and itself.

In simple words, if there is a nontrivial subspace of the representation that transforms in a self-contained way under the algebra elements, then the representation is reducible. Basically, it contains a more fundamental building block (the projection of itself on the mentioned subspace). Irreducible representations (irreps) are the ones which don't contain a more fundamental building block in themselves. Thus, they are themselves the fundamental building blocks of the representation theory and are of primary interest for those who wish to classify finite-dimensional representations of a given Lie algebra.

For every Lie algebra there is a special representation called the *adjoint*. It acts on the algebra itself by

$$\mathrm{ad} a \left|b\right\rangle = \left|[a,b]\right\rangle.$$

It is straightforward that the dimension of the adjoint representation is equal to the dimension of the algebra.

Enveloping algebra and Casimirs

For any Lie algebra $\mathfrak{g}$ we can define a *universal enveloping algebra*. It is an associative algebra of polynomials of the Lie algebra generators, for which the following holds:

$$\forall a, b \in \mathfrak{g}: \quad ab - ba = [a, b].$$

Here the associated product is taken in the enveloping algebra, while on the right hand side we encounter the Lie algebra bracket.

The particularly interesting for the representation theory are the special elements of the enveloping algebra called the Casimir operators which have the following property: they commute with all elements of the Lie algebra (in the enveloping algebra):

$$C: \quad \forall a \in \mathfrak{g}: \quad Ca = aC.$$

The exact form of Casimirs depends on the Lie algebra. It is important to keep in mind that Casimirs live in the universal enveloping algebra, not in the original Lie algebra. In general Casimirs are polynomials over the Lie algebra generators.

Casimirs are particularly important for representation theory because of the theorem called the Schur's lemma:

Schur's lemma: suppose we have two finite-dimensional irreducible representations ρ_1 and ρ_2 of the same finite-dimensional Lie algebra $\mathfrak{g}$ on two linear spaces V_1 and V_2 respectively and a linear operator $f : V_1 \to V_2$ which preserves the structure of the algebra, meaning that

$$f\rho_1 = \rho_2 f.$$

Then,

1. f is either a zero map (which maps any element of V_1 to zero), or an *isomorphism*.
2. In case $V_1 = V_2$ and $\rho_1 = \rho_2$, then $f = \lambda \cdot \mathbb{I}$ is a multiplication on some number $\lambda \in \mathbb{K}$.

As usual, I don't bother writing proofs of theorems in my notes. These can easily be found on the internet, and the purpose of this note is to give an introduction to the subject and state the most important results.

Now consider a representation ρ of some Lie algebra $\mathfrak{g}$. The key fact is that ρ defines naturally a representation of the universal enveloping algebra because matrices can be multiplied. Hence the Casimir can be represented as an $r \times r$ matrix f on the representation space V. Since it by definition commutes with the Lie algebra, the Schur's lemma is applicable here. We end up with an assertion of f to be a multiplication by some number λ.

A consequence of Schur's lemma: for any irrep ρ the Casimir operator is represented by a multiplication by some number λ. Thus, irreducibles can be labeled by the values of Lie algebra Casimirs.

Representation theory of $\mathfrak{su}(2)$

The $\mathfrak{su}(2)$ Lie algebra

The $\mathfrak{su}(2)$ Lie algebra is a simplest nontrivial example of the Lie algebra. It is 3-dimensional and defined by the following Lie bracket relations:

$$[t_\alpha, t_\beta] = \varepsilon_{\alpha\beta\gamma} t_\gamma.$$

These of course holds only in the basis used for defining the algebra, because $\varepsilon_{\alpha\beta\gamma}$ doesn't have the proper transformation properties like the ones structure constants have.

The definition can be rewritten in a more explicit form:

$$[x, y] = z$$

$$[y, z] = x$$

$$[z, x] = y$$

for the three algebra generators $\{x, y, z\}$.

Interesting fact: the $\mathfrak{su}(2)$ Lie algebra is equivalent to the $\mathfrak{so}(3)$ Lie algebra of the rotation group in 3 spatial dimensions:

$$\mathfrak{su}(2) \sim \mathfrak{so}(3).$$

We will see a lot of these "coincidental" equivalence relations between low-dimensional algebras. The classification theory of Lie algebras by Dynkin (covered later in this post) explains their origins.

Another interesting fact: $\mathfrak{su}(2)$ algebra is associative and is related to the vector multiplication in 3 dimensions. In fact, it exactly resembles the structure of vector multiplication for the three basis vectors $\{x, y, z\}$.

The quadratic Casimir of $\mathfrak{su}(2)$ is known from the quantum-mechanical theory of the angular momentum: it is equal to

$$C = x^2 + y^2 + z^2.$$

It is easy to show with a piece of algebra that C commutes with $\mathfrak{su}(2)$ in its enveloping algebra:

$$Cx - xC = x^3 + yyx + zzx - x^3 - xyy - xzz =$$

$$= yyx - xyy + zzx - xzz.$$

$$yyx - xyy = yyx - yxy + yxy - xyy = y(yx - xy) + (yx - xy)y =$$

$$= y[y, x] + [y, x]y = y(-z) + (-z)y = -(yz + zy);$$

$$zzx - xzz = zzx - zxz + zxz - xzz = z(zx - xz) + (zx - xz)z =$$

$$= z[z, x] + [z, x]z = zy + yz;$$

$$Cx - xC = -(yz + zy) + zy + yz = 0.$$

Analogously, we can show that

$$Cx - xC = Cy - yC = Cz - zC = 0.$$

This is sufficient to constatate that C is a Casimir of $\mathfrak{su}(2)$ and the irreps of $\mathfrak{su}(2)$ are labeled by the value of C (according to the consequence of the Schur's lemma).

The $\mathfrak{su}(2) \sim \mathfrak{sl}(2)$ relation and canonical basis

Consider another 3-dimensional Lie algebra $\mathfrak{sl}(2)$ which is defined by the Lie brackets

$$[h, e] = e$$

$$[h, f] = -f$$

$$[e, f] = 2h$$

for the three generators $\{e, f, h\}$.

I claim that $\mathfrak{su}(2) \sim \mathfrak{sl}(2)$, that is that the two algebras are equivalent (related by an isomorphism). In order to support my claim I have to provide the isomorphism between the vector spaces and work out the commutation relations in order to prove that it preserves the structure of the Lie algebra:

$$\begin{cases} e = ix - y \\ f = ix + y \\ h = iz \end{cases}$$

The proof that this is in fact an isomorphism of algebras:

$$[h, e] = [iz, ix - y] = -[z, x] - i[z, y] = -y - i(-x) = ix - y = e$$

$$[h, f] = [iz, ix + y] = -[z, x] + i[z, y] = -y + i(-x) = -ix - y = -f$$

$$[e, f] = [ix - y, ix + y] = i[x, y] - i[y, x] = 2i[x, y] = 2iz = 2h$$

This in a typical example of an important fact: the Lie algebras with different but related by a change of basis structure constants are equivalent.

Raising and lowering operators

Consider an r-dimensional irreducible representation ρ of $\mathfrak{su}(2) \sim \mathfrak{sl}(2)$ on the representation space V. Choose an eigenvector $|\Psi_m\rangle$ of $H = \rho(h)$ in V with eigenvalue m:

$$H |\Psi_m\rangle = m |\Psi_m\rangle .$$

We would like to exam the vectors

$$|\Psi_+\rangle = E |\Psi_m\rangle = \rho(e) |\Psi_m\rangle ,$$
$$|\Psi_-\rangle = F |\Psi_m\rangle = \rho(f) |\Psi_m\rangle .$$

In particular, we want to see how H acts on them. Consider the following calculation:

$$H |\Psi_+\rangle = HE |\Psi_m\rangle = (EH + [H, E]) |\Psi_m\rangle = (EH + E) |\Psi_m\rangle =$$

$$= (Em + E) |\Psi_m\rangle = (m + 1) |\Psi_+\rangle ,$$

$$H |\Psi_-\rangle = HF |\Psi_-\rangle = (FH + [F, H]) |\Psi_m\rangle = (FH - F) |\Psi_m\rangle =$$
$$(Fm - F) |\Psi_m\rangle = (m - 1) |\Psi_-\rangle .$$

We conclude that $|\Psi_\pm\rangle$ are also eigenvectors of H with eigenvalues $m \pm 1$. In general, we conclude that

$$|\Psi_+\rangle = A_m |\Psi_{m+1}\rangle ,$$

$$|\Psi_-\rangle = B_{m-1} |\Psi_{m-1}\rangle$$

for some numeric coefficients A_m and B_m.

We see that $E = \rho(e)$ and $F = \rho(f)$ act as raising and lowering operators on any irrep of $\mathfrak{su}(2)$: they increase and decrease the eigenvalue of $H = \rho(h)$ by one respectively. We arrive at the description of $\mathfrak{su}(2)$ irreps through the spectral decomposition of H.

As we will see later, this situation is not unique to $\mathfrak{su}(2)$. We will learn how to find analogues of the level operator H and raising/lowering operators E, F in more complex Lie algebras.

Recursive relations

The last commutation relation that we haven't used is

$$[E, F] = 2H.$$

Lets apply this to the vector $|\Psi_m\rangle$:

$$[E, F] |\Psi_m\rangle = EF |\Psi_m\rangle - FE |\Psi_m\rangle ;$$
$$EF |\Psi_m\rangle = EB_{m-1} |\Psi_{m-1}\rangle = A_{m-1}B_{m-1} |\Psi_m\rangle ;$$
$$FE |\Psi_m\rangle = FA_m |\Psi_{m+1}\rangle = A_m B_m |\Psi_m\rangle ;$$
$$2H |\Psi_m\rangle = 2m |\Psi_m\rangle ;$$
$$(A_{m-1}B_{m-1} - A_m B_m) |\Psi_m\rangle = 2m |\Psi_m\rangle .$$

This gives the recursive relation for the numeric coefficients A_m and B_m:

$$A_{m-1}B_{m-1} - A_m B_m = 2m.$$

Another relation can be deriving by considering unitarity of the irrep. Suppose that $X = \rho(x)$, $Y = \rho(y)$ and $Z = \rho(z)$ can be represented as antihermitian matrices (this is allowed by the commutation relations between them). Then

$$E^\dagger = (iX - Y)^\dagger = iXY = F,$$

$$F^\dagger = (iX + Y)^\dagger = iX - Y = E,$$
$$H^\dagger = (iZ)^\dagger = iZ = H.$$

Consequently, supposing that the eigensystem of H is properly normalized, the following holds:

$$\langle\Psi_{m+1}| E |\Psi_m\rangle = A_m \langle\Psi_{m+1}| \Psi_{m+1}\rangle = A_m,$$

$$\langle\Psi_{m+1}| F^\dagger |\Psi_m\rangle = \langle\Psi_m| F |\Psi_{m+1}\rangle^* = B_m^* \langle\Psi_m| \Psi_m\rangle^* = B_m^*,$$
$$F^\dagger = E \quad \Longrightarrow \quad A_m = B_m^*.$$

The recursive relation then becomes

$$|A_{m-1}|^2 - |A_m|^2 = 2m.$$

Solving the recursive relations

We've come close to the classification of all finite-dimensional irreducible representations of $\mathfrak{su}(2)$.

We are considering finite-dimensional representations, thus there must be an eigenvector of H with maximal possible eigenvalue, and another one with minimal possible eigenvalue. We will call these eigenvalues

$$m_{\min}, \quad m_{\max}.$$

Since $E |\Psi_m\rangle \sim |\Psi_{m+1}\rangle$ and there could be no eigenvalue greater than $m_{\max}$, we can only conclude that

$$E |\Psi_{m_{\max}}\rangle = F |\Psi_{m_{\min}}\rangle = |0\rangle .$$

Or, in terms of the coefficients,

$$A_{m_{\min}-1} = A_{m_{\max}} = 0.$$

What we would like to do is to sum our recursive relation for m from some $m_0 + 1$ to $j = m_{\max}$:

$$\sum_{m_0+1}^{j} \left(|A_{m-1}|^2 - |A_m|^2\right) = 2 \sum_{m_0+1}^{j} m.$$

On the right-hand side there is an arithmetic progression, which can be easily summed:

$$\sum_{m_0+1}^{j} \left(|A_{m-1}|^2 - |A_m|^2\right) = (j - m_0)(m_0 + j + 1) .$$

On the left-hand side the terms in the sum cancel each other except for the boundary terms:

$$|A_{m_0}|^2 - |A_j|^2 = |A_{m_0}|^2 = (j - m_0)(m_0 + j + 1).$$

This gives us the expression for the coefficients:

$$|A_m|^2 = (j - m)(j + m + 1).$$

As expected, the two zeros of this expression correspond to $m_{\text{max}} = j$ and $m_{\text{min}} - 1 = -(j + 1)$. Thus we conclude that

$$m_{\text{min}} = -m_{\text{max}}.$$

Finally, since all m differ by one, the expression $m_{\text{max}} - m_{\text{min}}$ must be a non-negative integer:

$$m_{\text{max}} - m_{\text{min}} = 2j \in \mathbb{N}_+.$$

Classification of $\mathfrak{su}(2)$ irreps

We conclude that finite-dimensional irreducibles of $\mathfrak{su}(2)$ are labeled by a nonnegative integer j called the *spin* of the irrep. The spin-j irrep has eigenvectors of H with eigenvalues all the way from $-j$ to j. That gives the dimensionality of the spin-j irrep:

$$r = \dim\rho = j - (-j) + 1 = 2j + 1.$$

Note the two important facts:

1. If we had several series of eigenvectors of H with all series consisting of eigenvalues which differ by one but with an arbitrary shift between the eigenvalues from different series, then our initial assumption that we are dealing with an irreducible representation is wrong. The independent series describe different irreps and our (reducible) representation under consideration is a direct sum of these irreps.
2. We are only considering finite-dimensional representations here. The classification of infinite-dimensional representations is different.

Finally, let's reproduce some of the low-dimensional irreps of $\mathfrak{su}(2)$ explicitly.

The spin-0 irrep

The spin-0 (trivial) irrep is 1-dimensional, with

$$H = E = F = X = Y = Z = (0).$$

All commutation relations trivially hold.

The Casimir operator is also zero.

The spin-1/2 (fundamental) irrep

The 2-dimensional spin-1/2 irrep is called fundamental for reasons that will become clear later. Lets construct it.

1. First we have to calculate the values of the coefficients A_m:

$$A_m = e^{i\varphi_m}\sqrt{(j - m)(j + m + 1)},$$
$$B_m = e^{-i\varphi_m}\sqrt{(j - m)(j + m + 1)},$$

where φ_m is an arbitrary phase that does not influence the resulting irrep (the irreps with different φ_m are related through isomorphisms). We substitute $j = 1/2$ and the two relevant values of m which are $m = \pm 1/2$ and take $\varphi_{-1/2} = 0$ for convenience:

$$A_{1/2} = B_{1/2} = 0,$$
$$A_{-1/2} = B_{-1/2} = 1,$$

2. Recall that it means that

$$E\left|\Psi_{-1/2}\right\rangle = A_{-1/2}\left|\Psi_{1/2}\right\rangle = \left|\Psi_{1/2}\right\rangle,$$
$$F\left|\Psi_{1/2}\right\rangle = B_{-1/2}\left|\Psi_{-1/2}\right\rangle = \left|\Psi_{-1/2}\right\rangle.$$

3. Take the basis vectors to be

$$\left|\Psi_{1/2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|\Psi_{-1/2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix representation of E and F follows immediately:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. Remember how $H\left|\Psi_m\right\rangle = m\left|\Psi_m\right\rangle$? This immediately gives the matrix representation of H:

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

5. We can now recover the matrices for the basis corresponding to the definition of $\mathfrak{su}(2)$ which is $\{X, Y, Z\}$:

$$X = \frac{E+F}{2i} = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix},$$
$$Y = \frac{F-E}{2} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$
$$Z = \frac{H}{i} = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}.$$

I leave it to the reader to show that $\{E, F, H, X, Y, Z\}$ satisfy the commutation relations of $\mathfrak{sl}(2) \sim \mathfrak{su}(2)$.

We have just calculated the explicit formulas for the spin-$1/2$ irreducible representation of $\mathfrak{su}(2)$. It is exactly the well-known basis given by the generators

$$t_\alpha = -\frac{i}{2}\sigma_\alpha,$$

where σ_α are the three Pauli matrices. Indeed,

$$X = -\frac{i}{2}\sigma_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = t_1,$$
$$Y = -\frac{i}{2}\sigma_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = t_2,$$
$$Z = -\frac{i}{2}\sigma_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = t_3.$$

Thus the spin-$1/2$ irrep of $\mathfrak{su}(2)$ is given by the Pauli matrices.

The Casimir operator can be evaluated explicitly:

$$C = X^\dagger X + Y^\dagger Y + Z^\dagger Z = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} = \frac{3}{4} \cdot \mathbb{I}.$$

You might wonder why we use Hermissian conjugation in the Casimir. The is actually because the enveloping algebra is a C^* algebra. For the purposes of this introduction it is just not important.

The generic formula for the Casimir reads

$$C_j = j(j+1) \cdot \mathbb{I},$$

which coincides with the value $3/4$ for spin-$1/2$.

The spin-1 (adjoint) irrep

The next irrep in our series is the 3-dimensional spin-1 irrep. The fact that its dimensionality equals the dimensionality of the Lie algebra is not accidental: it is easy to prove that it is isomorphic to the adjoint representation of $\mathfrak{so}(3) \sim \mathfrak{sl}(2) \sim \mathfrak{su}(2)$. Lets construct it.

1. First we have to calculate the values of the coefficients A_m:

$$A_m = e^{i\varphi_m}\sqrt{(j-m)(j+m+1)},$$

$$B_m = e^{-i\varphi_m}\sqrt{(j-m)(j+m+1)},$$

   We substitute $j = 1$ and the three relevant values of m which are $m \in \{-1, 0, 1\}$ and take $\varphi_{\dots} = 0$ for convenience:

$$A_1 = B_1 = 0,$$

$$A_0 = B_0 = \sqrt{2}$$

$$A_{-1} = B_{-1} = \sqrt{2},$$

2. Recall that it means that

$$E\,|\Psi_{-1}\rangle = A_{-1}\,|\Psi_0\rangle = \sqrt{2}\,|\Psi_0\rangle\,,$$
$$E\,|\Psi_0\rangle = A_0\,|\Psi_1\rangle = \sqrt{2}\,|\Psi_1\rangle\,,$$
$$F\,|\Psi_1\rangle = A_0\,|\Psi_0\rangle = \sqrt{2}\,|\Psi_0\rangle\,,$$
$$F\,|\Psi_0\rangle = A_{-1}\,|\Psi_{-1}\rangle = \sqrt{2}\,|\Psi_{-1}\rangle\,.$$

3. Take the basis vectors to be

$$|\Psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\Psi_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\Psi_{-1}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

   The matrix representation of E and F follows immediately:

$$E = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

4. The matrix representation of H is straightforward:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

5. We can now recover the matrices for the basis corresponding to the definition of $\mathfrak{su}(2)$ which is $\{X, Y, Z\}$:

$$X = \frac{E+F}{2i} = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$Y = \frac{F-E}{2} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$Z = \frac{H}{i} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

The reader can check that the commutation relations hold. This is related to the standard $\mathfrak{so}(3)$ basis (the adjoint basis of the algebra)

$$X \to \theta_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$Y \to \theta_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$Z \to \theta_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

through a change of basis (find one).

The Casimir operator can be evaluated explicitly:

$$C = X^\dagger X + Y^\dagger Y + Z^\dagger Z = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot \mathbb{I},$$

which coincides with the general formula

$$C_j = j(j+1) \cdot \mathbb{I}.$$

Tensor products and intertwiners

This section can be skipped on the first read.

To any Lie algebra we associate a recoupling theory. It tells us how tensor products of irreps (which are reducible in general) can be expanded in direct sums of irreducibles.

For $\mathfrak{su}(2)$ the general formula is available:

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2| + 1 \oplus \cdots \oplus (j_1 + j_2)\,.$$

All irreps between spins $|j_1 - j_2|$ and $(j_1 + j_2)$ enter in this series either one time or zero times (depending on whether the spin is integer or half-integer). For example, consider

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

This formula tells us that the tensor product of two 2-dimensional fundamental spin-$1/2$ irreps is equivalent to the direct sums of trivial and adjoint.

A special kind of tensors are the *intertwining tensors* or *intertwiners*. These have k indices, each one in different (!) representations of the same Lie algebra. The defining property is that the tensor is invariant under any algebra element acting simultaneously on all of the indices.

For the case of $\mathfrak{su}(2)$ it is easy to find intertwiners from the recoupling theory. For example, if I wanted to find 3-valent intertwiners with two irreps being the spin-$1/2$ and another spin-$1/2$, I would only find the intertwiners between $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$ because of the tensor product $\frac{1}{2} \otimes \frac{1}{2}$.

Generic representation theory

We would now like to generalize the results from the previous section to the general case of an arbitrary semisimple Lie algebra.

Cartan subalgebra

First we need a generalization of the element h of $\mathfrak{sl}(2)$ which we've used for the spectral decomposition of irreps. In the general case it is played by the Cartan subalgebra.

The *Cartan subalgebra* is defined to be the linear span of the largest subset of intercommuting generators. That is, the Lie brackets of the elements of the Cartan subalgebra vanish.
The dimensionality r of the Cartan subalgebra is called the *rank* of the Lie algebra.

For example, for the case of $\mathfrak{su}(2)$ the rank is equal to 1 and any generator can be chosen as a basis of the 1-dimensional Cartan subalgebra. We've chosen h, but actually this choice is purely conventional.

Lets illustrate the concept of Cartan subalgebra on a less trivial example: the 8-dimensional Lie algebra $\mathfrak{su}(3)$. Instead of writing down the commutation relations (of which there are 28) it is more convenient to give a *defining representation* of $\mathfrak{su}(3)$, that is, a representation, which we use to calculate the structure constants and define the algebra. The defining representation of $\mathfrak{su}(3)$ can be chosen arbitrarily, but the most convenient is one of its two fundamental representations (the terminology will become clear later) given by the 8 Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

The Cartan subalgebra is formed by the linear span of $\{\lambda_3, \lambda_8\}$ (it can be shown that they intercommute while all the other basis elements don't commute with both of them at the same time). Thus, the rank of the 8-dimensional Lie algebra $\mathfrak{su}(3)$ is 2.

Cartan-Weyl basis

We denote the generators of the Cartan subalgebra by c_i with i running through $1..r$.

The purpose of this section is to generalize the notion of raising/lowering operators. Remember how in the $\mathfrak{su}(2)$ case we had

$$[h, e] = e,$$

$$[h, f] = -f?$$

Well, in the general case this can be mimicked by requiring the remaining generators to satisfy the eigenvector equation for the adjoint action of the Cartan subalgebra generators:

$$[c_i, e_{\vec{\alpha}}] = \alpha_i e_{\vec{\alpha}}.$$

This depends on a collection of *roots* — vectors in the dual to the Cartan subalgebra space denoted by $\vec{\alpha}$. The coordinates with respect to the basis of Cartan generators are denoted by α_i.

To each root $\vec{\alpha}$ we associate an element $e_{\vec{\alpha}}$ of the Lie algebra chosen as an eigenvector of the adjoint action of Cartans with eigenvalues being exactly the coordinates of the root.

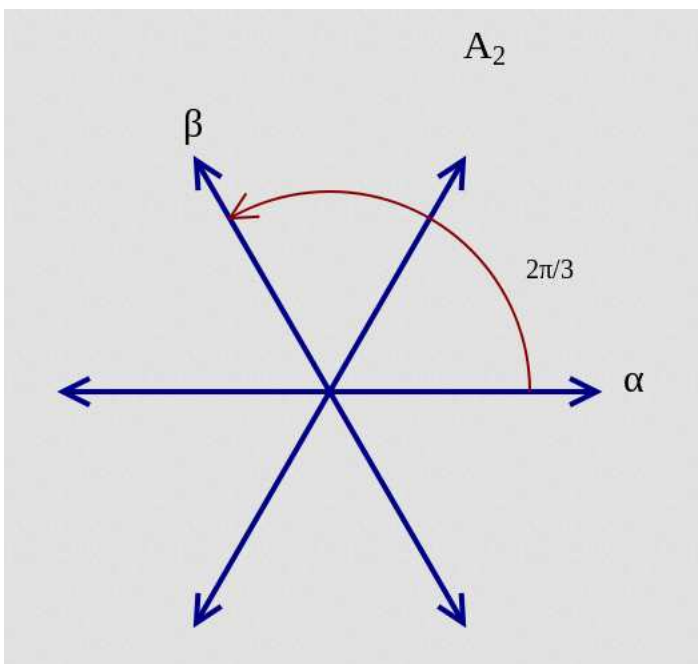
The Cartan-Weyl basis is most convenient for representation theory. It consists of two parts:

1. The generators of the Cartan subalgebra labeled by c_i with i running through $1..r$.
2. The root elements $e_{\vec{\alpha}}$ associated to roots $\vec{\alpha}$ in the root system. The defining equation for the Weyl part of the Cartan-Weyl basis is
$$[c_i, e_{\vec{\alpha}}] = \alpha_i e_{\vec{\alpha}}.$$

An example: the A2 root system

Lets try to calculate the Cartan-Weyl basis for $\mathfrak{su}(3)$. We have rank 2, thus the Cartan subalgebra and the root system (which is dual to the Cartan subalgebra) are 2-dimensional. The roots are represented by 2-dimensional vectors.

How many roots are there? Well, the total number of generators is the dimensionality of the algebra, which is 8. Two generators belong to the Cartan subalgebra. The remaining 6 generators form the 6 roots of the A2 root system:



The picture provides some useful information. For example, just from looking at the picture I know the following commutation relations:

$$[c_1, e_{\vec{\alpha}}] = e_{\vec{\alpha}},$$
$$[c_2, e_{\vec{\alpha}}] = 0,$$
$$\left[c_1, e_{\vec{\beta}}\right] = -\frac{1}{2} e_{\vec{\beta}},$$
$$\left[c_2, e_{\vec{\beta}}\right] = \frac{\sqrt{3}}{2} e_{\vec{\beta}},$$

These are, of course, special cases of the defining formula

$$[c_i, e_{\vec{\alpha}}] = \alpha_i e_{\vec{\alpha}}.$$

The roots $\vec{\alpha}$ and $\vec{\beta}$ are called *simple*: they form the basis in the root system.

Raising and lowering with roots

Consider a finite-dimensional irreducible representation ρ of the semisimple Lie algebra $\mathfrak{g}$. We use the Cartan generators as operators for the spectral decomposition of the representations into 1-dimensional eigenspaces of Cartan generators.

We denote by $|\Psi_{\vec{a}}\rangle$ an eigenspace of the Cartan subalgebra defined by a *weight vector* $\vec{a}$ on the root space. The weight vectors are analogous to the number m in the case of $\mathfrak{su}(2)$. The defining equation is

$$C_i |\Psi_{\vec{a}}\rangle = \rho(c_i) |\Psi_{\vec{a}}\rangle = a_i |\Psi_{\vec{a}}\rangle .$$

We want to understand how the root vector $e_{\vec{\alpha}}$ acts on this state. For this, by analogy with the $\mathfrak{su}(2)$ case, we examine the state

$$|\Phi\rangle = E_{\vec{\alpha}} |\Psi_{\vec{a}}\rangle = \rho(e_{\vec{\alpha}}) |\Psi_{\vec{a}}\rangle .$$

We hope that just like in the $\mathfrak{su}(2)$ case this will turn out to be the eigenstate of Cartans. This is indeed true:

$$C_i |\Phi\rangle = C_i E_{\vec{\alpha}} |\Psi_{\vec{a}}\rangle = (E_{\vec{\alpha}} C_i + [C_i, E_{\vec{\alpha}}]) |\Psi_{\vec{a}}\rangle =$$

$$= (a_i E_{\vec{\alpha}} + \alpha_i E_{\vec{\alpha}}) |\Psi_{\vec{a}}\rangle = (a_i + \alpha_i) |\Phi\rangle .$$

When a root vector $e_{\vec{\alpha}}$ acts on the state with weight $\vec{a}$, the resulting state is also an eigenstate of the Cartan subalgebra with resulting weight

$$\vec{b} = \vec{a} + \vec{\alpha}.$$

Consider the trivial case of $\mathfrak{su}(2)$ again. Here the rank is 1 and we have two roots:

$$e_1 = e,$$

$$e_{-1} = f.$$

The action of these roots on the eigenstate of the Cartan $c_1 = h$ adds 1 and -1 to the weight. This is exactly raising/lowering of the eigenvalue of h!

Thus, the generalization of the raising/lowering is vector addition of the root vector and the weight vector in the linear space of the root system, dual to the Cartan subalgebra.

Weyl-Weyl Lie brackets

We have three types of Lie brackets in the algebra expanded over the Cartan-Weyl basis:

1. Cartan-Cartan Lie brackets are all zero: $[c_i, c_j] = 0$. It is the definition of the Cartan subalgebra.
2. The Cartan-Weyl Lie brackets are determined by roots. It is hard to overemphasize the importance of this formula, so I am going to write it down another time (probably third of fourth): $[c_i, e_{\vec{\alpha}}] = \alpha_i e_{\vec{\alpha}}$.
3. The Weyl-Weyl Lie brackets, or Lie brackets between roots are those which we haven't considered yet.

In order to obtain a generalization of the recursion relations we have to consider the Lie brackets between roots. It is the subject of the present section. So lets work out these brackets. We have (by the Jacobi identity)

$$\left[c_i, \left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right]\right] + \left[e_{\vec{\alpha}}, \left[e_{\vec{\beta}}, c_i\right]\right] + \left[e_{\vec{\beta}}, [c_i, e_{\vec{\alpha}}]\right] = 0.$$

By substituting the formula for the Cartan-Weyl brackets we have

$$\left[c_i, \left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right]\right] - \beta_i \left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right] + \alpha_i \left[e_{\vec{\beta}}, e_{\vec{\alpha}}\right],$$

$$\left[c_i, \left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right]\right] = (\alpha_i + \beta_i) \left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right].$$

Thus, $\left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right]$ is proportional to $e_{\vec{\alpha}+\vec{\beta}}$ with some yet undetermined coefficient. There are two remarks that I would like to give:

1. Note that $\left[e_{\vec{\alpha}}, e_{\vec{\beta}}\right]$ changes sign if we replace $\alpha \leftrightarrow \beta$, while $e_{\vec{\alpha}+\vec{\beta}}$ seemingly doesn't. This is actually absolutely normal, because we haven't considered the proportionality coefficient. This coefficient changes sign.
2. Note how this is similar to the action of roots on states with weights in some representation ρ. This is not accidental, in fact, *roots are weights in the adjoint representation* where the action of algebra elements is given by the Lie bracket.

Killing form and normalization

The coefficient can be calculated for a special case of $\vec{\alpha}+\vec{\beta}=0$ (corresponding to rasing/lowering). For this case, $[e_{\vec{\alpha}}, e_{-\vec{\alpha}}]$ is proportional for $e_{\vec{0}}$, but there is no root with $\vec{0}$, since if there was, it would commute with all of the Cartan generators and we would include it in the Cartan subalgebra. We conclude that $[e_{\vec{\alpha}}, e_{-\vec{\alpha}}]$ lies in the Cartan subalgebra:

$$[e_{\vec{\alpha}}, e_{-\vec{\alpha}}] = q_i c_i.$$

The coefficients q_i depend on the normalization of the basis, but we can fix this by evaluating the Killing form on the Lie algebra:

$$\forall a, b \in \mathfrak{g}: \quad g(a,b) = \mathrm{tr}\left[\mathrm{ad}a \cdot \mathrm{ad}b\right].$$

The Killing form is by definition equal to the trace of products of adjoint-representation martices associated with Lie algebra elements. This is why it is often called the invariant trace.

We impose the following normalization condition on the Cartan generators and on roots:

$$g(c_i, c_j) = \delta_{ij},$$
$$g(e_{\vec{\alpha}}, e_{\vec{\beta}}) = \delta_{\vec{\alpha},-\vec{\beta}}.$$

The coefficients q_i can then be calculated:

$$g\left(c_i, [e_{\vec{\alpha}}, e_{-\vec{\alpha}}]\right) = q_j g(c_i, c_j) = q_i.$$

But on the other hand,

$$g\left(c_i, [e_{\vec{\alpha}}, e_{-\vec{\alpha}}]\right) = \mathrm{tr}\left([\mathrm{ad}c_i, \mathrm{ad}e_{\vec{\alpha}}] \cdot \mathrm{ad}e_{-\vec{\alpha}}\right) = \alpha_i \mathrm{tr}\left(\mathrm{ad}e_{\vec{\alpha}} \cdot \mathrm{ad}e_{-\vec{\alpha}}\right) = \alpha_i.$$

Thus, $q_i = \alpha_i$ and we have

$$[e_{\vec{\alpha}}, e_{-\vec{\alpha}}] = \alpha_i c_i.$$

Generic recursion relations

The next thing to do is to introduce the coefficients analogous to A_m and B_m from the $\mathfrak{su}(2)$ case:

$$E_{\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle = A_{\vec{\alpha},\vec{a}} \left|\Psi_{\vec{a}+\vec{\alpha}}\right\rangle.$$

Note that there is no need for the letter B: the raising and lowering operators are on the equal footing, the canonical pairs corresponding to the generators associated to the opposite roots.

By analogy with $\mathfrak{su}(2)$, we would like to derive two recursive relations for $A_{\vec{\alpha},\vec{a}}$. By acting on $\left|\Psi_{\vec{a}}\right\rangle$ with $E_{-\vec{\alpha}}E_{\vec{\alpha}} - E_{\vec{\alpha}}E_{-\vec{\alpha}}$ we get:

$$E_{-\vec{\alpha}}E_{\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle = A_{\vec{a},\vec{\alpha}} E_{-\vec{\alpha}} \left|\Psi_{\vec{a}+\vec{\alpha}}\right\rangle = A_{\vec{a},\vec{\alpha}} A_{\vec{a}+\vec{\alpha},-\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle,$$
$$E_{\vec{\alpha}}E_{-\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle = A_{\vec{a},-\vec{\alpha}} E_{\vec{\alpha}} \left|\Psi_{\vec{a}-\vec{\alpha}}\right\rangle = A_{\vec{a},-\vec{\alpha}} A_{\vec{a}-\vec{\alpha},\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle,$$
$$E_{-\vec{\alpha}}E_{\vec{\alpha}} - E_{\vec{\alpha}}E_{-\vec{\alpha}} = \alpha_i C_i,$$
$$\left(A_{\vec{a},\vec{\alpha}} A_{\vec{a}+\vec{\alpha},-\vec{\alpha}} - A_{\vec{a},-\vec{\alpha}} A_{\vec{a}-\vec{\alpha},\vec{\alpha}}\right) \left|\Psi_{\vec{a}}\right\rangle = \alpha_i C_i \left|\Psi_{\vec{a}}\right\rangle = \alpha_i a_i \left|\Psi_{\vec{a}}\right\rangle.$$

We arrive at the following recursion relation:

$$A_{\vec{a},\vec{\alpha}} A_{\vec{a}+\vec{\alpha},-\vec{\alpha}} - A_{\vec{a},-\vec{\alpha}} A_{\vec{a}-\vec{\alpha},\vec{\alpha}} = \vec{\alpha} \cdot \vec{a}.$$

The second recursion relation is derived through unitarity (by analogy with the $\mathfrak{su}(2)$ case). We choose the Cartan generators to be hermitian and the root vectors to change the sign of the root upon conjugation:

$$C_i^\dagger = C_i,$$
$$E_{\vec{\alpha}}^\dagger = E_{-\vec{\alpha}}.$$

Then we have

$$\left\langle\Psi_{\vec{a}+\vec{\alpha}}\right| E_{\vec{\alpha}} \left|\Psi_{\vec{a}}\right\rangle = A_{\vec{a},\vec{\alpha}} \left\langle\Psi_{\vec{a}+\vec{\alpha}} \middle| \Psi_{\vec{a}+\vec{\alpha}}\right\rangle = A_{\vec{a},\vec{\alpha}};$$
$$\left\langle\Psi_{\vec{a}+\vec{\alpha}}\right| E_{-\vec{\alpha}}^\dagger \left|\Psi_{\vec{a}}\right\rangle = \left\langle\Psi_{\vec{a}}\right| E_{-\vec{\alpha}} \left|\Psi_{\vec{a}+\vec{\alpha}}\right\rangle^* = A^*_{\vec{a}+\vec{\alpha},-\vec{\alpha}} \left\langle\Psi_{\vec{a}} \middle| \Psi_{\vec{a}}\right\rangle = A^*_{\vec{a}+\vec{\alpha},-\vec{\alpha}};$$
$$E_{\vec{\alpha}} = E_{-\vec{\alpha}}^\dagger \quad \Longrightarrow \quad A_{\vec{a}+\vec{\alpha},-\vec{\alpha}} = A^*_{\vec{a},\vec{\alpha}}.$$

Combining the two relations, we obtain:

$$\left|A_{\vec{a}-\vec{\alpha},\vec{\alpha}}\right|^2 - \left|A_{\vec{a},\vec{\alpha}}\right|^2 = \vec{a} \cdot \vec{\alpha}.$$

Solving generic recursion relations

Just like in the $\mathfrak{su}(2)$ case we are looking for finite-dimensional irreducible representations of our Lie algebra. Thus there must be a minimal and maximal weights

$$\vec{a}_{\text{min}} = \vec{a} - q\vec{\alpha}, \quad \vec{a}_{\text{max}} = \vec{a} + p\vec{\alpha}$$

for any root $\vec{\alpha}$. By analogy with the $\mathfrak{su}(2)$ case we add the recursion relations for all weights $\vec{a}$ between $\vec{a}_{\text{min}}$ and $\vec{a}_{\text{max}}$ and obtain:

$$|A_{\vec{a}-q\vec{\alpha},\vec{\alpha}}|^2 = (p+q+1) \cdot \left(\vec{a} \cdot \vec{\alpha} + \vec{\alpha} \cdot \vec{\alpha} \cdot \frac{(p-q)}{2} \right).$$

For the lowest weight we have

$$|A_{\vec{a}-q\vec{\alpha},\vec{\alpha}}|^2 = 0$$

which has two solutions:

1. $p + q + 1 = 0$
2. $\frac{\vec{a}\cdot\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}} = \frac{q-p}{2}$

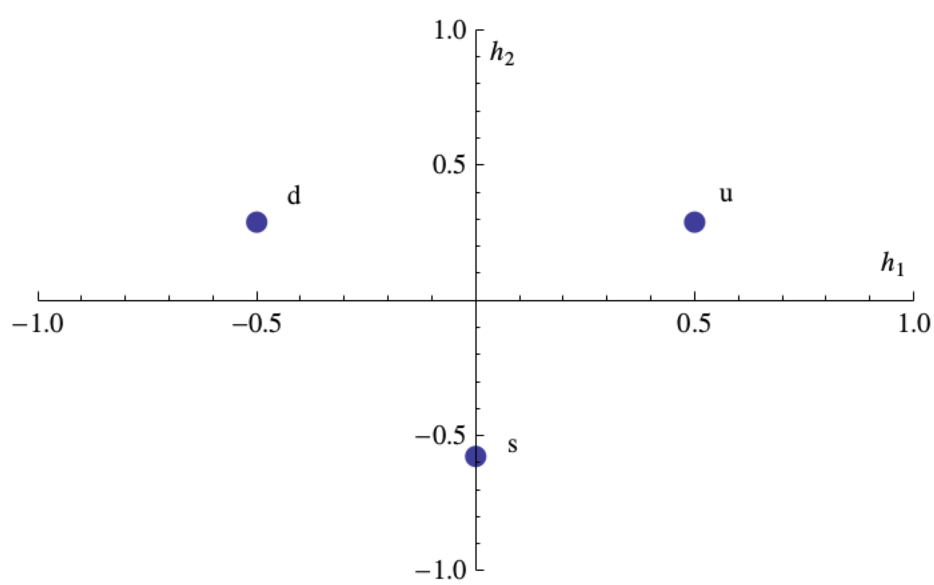
The first solution was there in the $\mathfrak{su}(2)$ case. It gives contribution from the weight $\vec{a}$ to the dimensionality of the irrep under consideration, just like $2j+1$ gives contribution to the dimensionality of the spin-j irrep of $\mathfrak{su}(2)$.

The second relation is particularly interesting. We will come to it later.

Classifying representations of Lie algebras

Finite-dimensional irreducible representations are classified by the highest weights of simple roots. They correspond to the *weight diagrams* — patterns on the root system with points connected by root vectors. The root system itself is a weight diagram of the adjoint representation.

As an example we give the weight diagram of the first fundamental representation of $\mathfrak{su}(3)$:



In the flavour $SU(3)$ model it corresponds to the three flavours of quarks ("u" for "up", "d" for "down", "s" for "strange"). It is worth mentioning that the modern $SU(3)$ theory is based on colors and the flavour symmetry is considered approximate.

The irrep is 3-dimensional and can be decomposed in three 1-dimensional subspaces which belong to the spectra of Cartan generators. For example, for the "u" quark

$$c_1 |u\rangle = \frac{1}{2} |u\rangle .$$

This can be read off from the x-coordinate of the "u" point.

The action of roots on these subspaces is given by the vector addition in the root space. The $e_{\vec{\alpha}}$ vector from the root diagram above, for example, takes $|d\rangle$ to $|u\rangle$ (with some coefficient), while the $e_{\vec{\beta}}$ vector takes $|s\rangle$ to $|d\rangle$ (also with some coefficient).

Just like we did with the fundamental irrep of $\mathfrak{su}(2)$ and Pauli matrices, the representation matrices can be constructed to give the Gell-Mann matrices. It is a long and tedious calculation, so I am going to omit it here, but carrying it out would certainly help to get more fluent in the subject.

In case of $\mathfrak{su}(3)$ another fundamental representation exists called the second fundamental representation. Its weight diagram can be obtained from the diagram for the first fundamental representation by rotating it by π.

Classification of semisimple Lie algebras

From the previous section we learned that any semisimple Lie algebra with rank r and dimensionality n is described by a r-dimensional root system consisting of $n - r$ roots. Of those, r roots are simple.

Dynkin diagrams

The strange second solution for the recursion relation can be rewritten for the adjoint representation. Weights become roots, and the solution reads:

$$\frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} = \frac{q - p}{2}$$

for all roots $\vec{\alpha}$ and $\vec{\beta}$. This can be interpreted as follows: the projection of any root on any other root has half-integer length.

A Dynkin root system is an r-dimensional root system consisting of r simple roots with the following properties:

1. Any two simple roots are linearly independent;
2. For any two roots $\vec{\alpha}$ and $\vec{\beta}$ the projection of one on the other has half-integer length;
3. The system is not reducible to a sum of subsystems orthogonal to each other.

We choose to label simple roots by nodes of a certain graph called the *Dynkin diagram.*

In order to satisfy the second requirement the angle between simple roots can only have the following values:

1. $\pi/2$ — in this case we don't draw an edge between the nodes
2. $2\pi/3$ — in this case we draw an edge between the nodes
3. $3\pi/4$ — in this case we draw a double edge between the nodes
4. $5\pi/6$ — in this case we draw a triple edge between the nodes
5. the cases when $\varphi < \pi/2$ can be treated by choosing another subset of simple roots

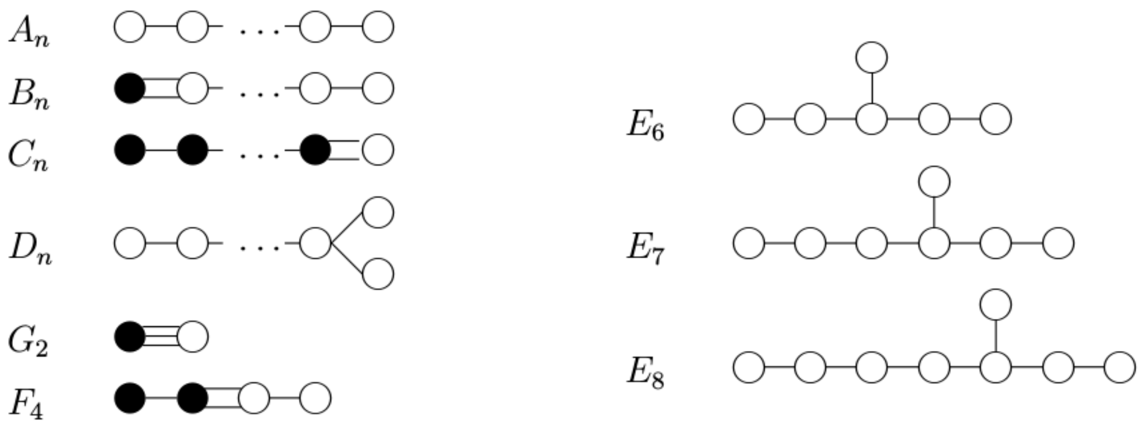
Also we would choose to paint the circle representing the node of the graph if the length of the root associated with this node is smaller than the lengths of nearby roots.

Semisimple Lie algebras

We've reduced the problem of classification of semisimple Lie algebras to simple combiniatorics. We have to enumerate all the graphs which give rise to acceptable Dynkin root systems.

These are given by the four infinite series A_n, B_n, C_n and D_n as well as by five exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8.

The Dynkin diagrams of these root system are drawn below.

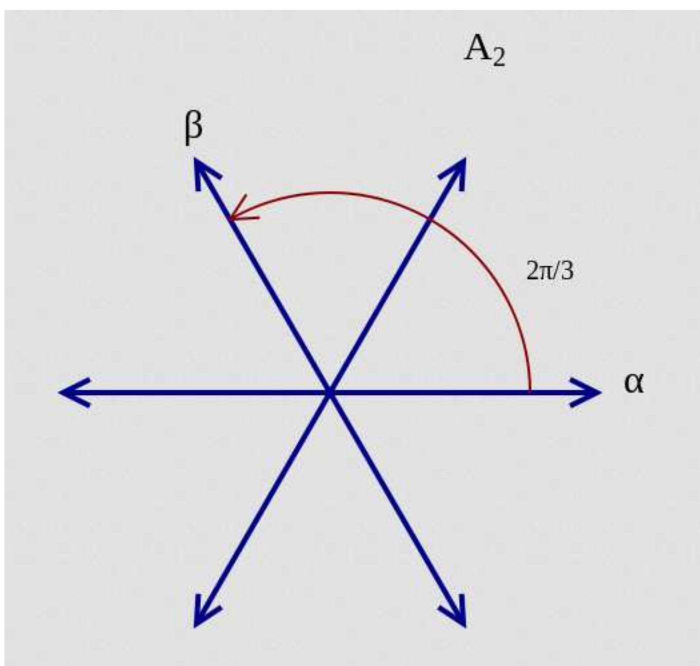


Notably, the exceptional Lie algebras have been a subject of theories of unification of fundamental forces. The $E_8 \times E_8$ is one of the possible gauge groups of heterotic superstrings. Also, E_8 has been used by Lisi to construct a (speculative) model for the unification of all forces. In the same manner, G_2 is used by Lisi to unify quarks with gluons in a single $\mathfrak{g}_2$-valued superconnection.

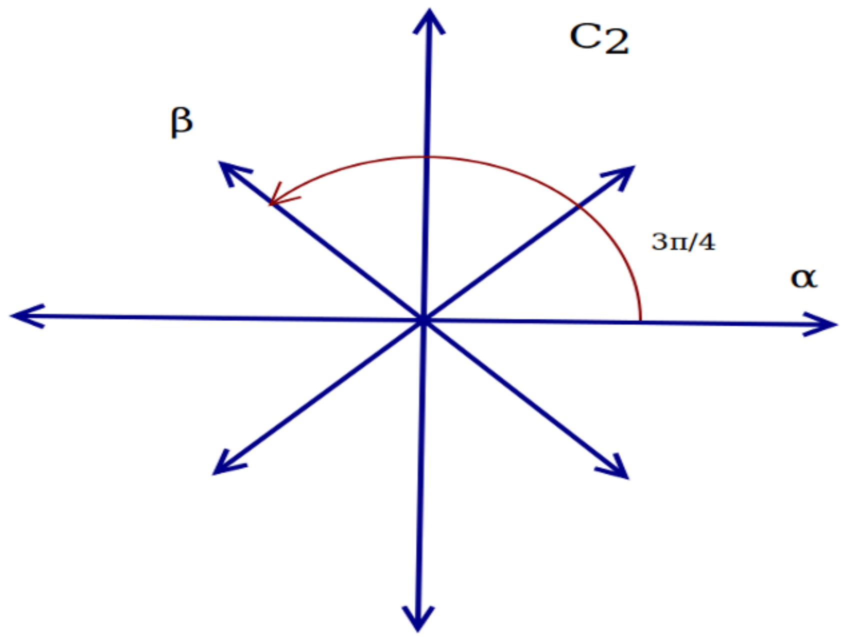
Rank-2 root systems

I include the pictures of the rank-2 root systems:

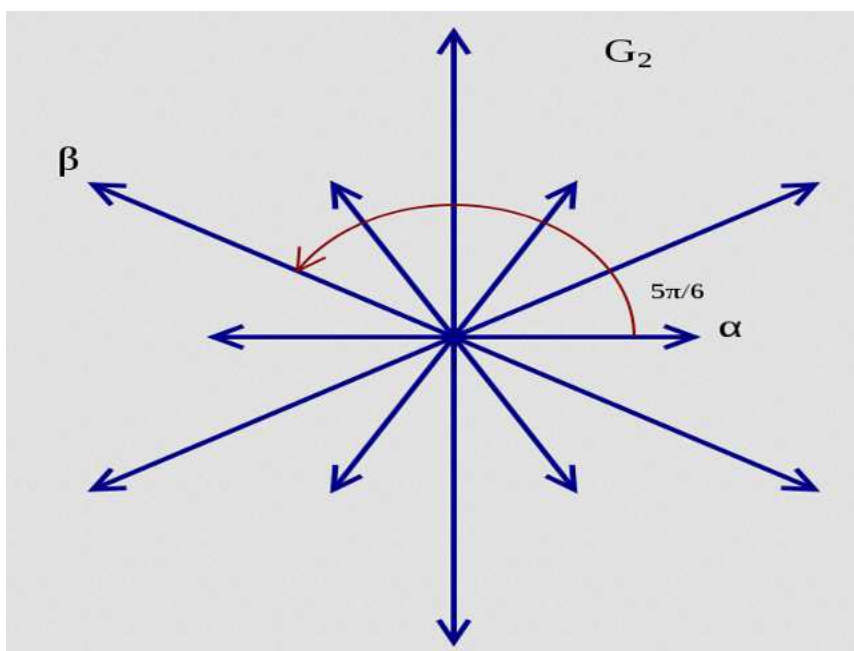
- We have already encountered the A_2 root system of the $\mathfrak{su}(3)$ Lie algebra:



- The $B_2 \sim C_2$ root system:



- D_2 is technically not a valid root system, as it is equal to the sum of two A_1 root systems. This is why $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as A_1 is the root system of $\mathfrak{su}(2) \sim \mathfrak{so}(3)$ and $D_2 = A_1 \cup A_1$ is the root system of $\mathfrak{so}(4)$.
- The G_2 root system:



Calculating root systems

I've written a simple Haskell script which can generate the coordinates of roots from the Dynkin diagram. It uses Wolfram Mathematica to perform symbolic computations. It is available on my github.

Examples of calculations:

For the (rank-1) A_1 root system (the $\mathfrak{su}(2)$ Lie algebra):

```
rootsLen = 2;
r[1] = {1};
r[2] = {-1};
```

For the (rank-2) A_2 root system (the $\mathfrak{su}(3)$ Lie algebra):

```
rootsLen = 6;
r[1] = {1, 0};
r[2] = {-1/2, Sqrt[3]/2};
r[3] = {-1, 0};
r[4] = {1/2, Sqrt[3]/2};
r[5] = {1/2, -Sqrt[3]/2};
r[6] = {-1/2, -Sqrt[3]/2};
```

For the (rank-4) D_4 root system (the one with the triality symmetry):

```
rootsLen = 24;
r[1] = {1, 0, 0, 0};
r[2] = {-1/2, Sqrt[3]/2, 0, 0};
r[3] = {-1/2, -1/(2*Sqrt[3]), Sqrt[2/3], 0};
r[4] = {-1/2, -1/(2*Sqrt[3]), -(1/Sqrt[6]), 1/Sqrt[2]};
r[5] = {-1, 0, 0, 0};
r[6] = {1/2, Sqrt[3]/2, 0, 0};
r[7] = {1/2, -1/(2*Sqrt[3]), Sqrt[2/3], 0};
r[8] = {1/2, -1/(2*Sqrt[3]), -(1/Sqrt[6]), 1/Sqrt[2]};
r[9] = {1/2, -Sqrt[3]/2, 0, 0};
r[10] = {1/2, 1/(2*Sqrt[3]), -Sqrt[2/3], 0};
r[11] = {1/2, 1/(2*Sqrt[3]), 1/Sqrt[6], -(1/Sqrt[2])};
r[12] = {-1/2, -Sqrt[3]/2, 0, 0};
r[13] = {-1/2, 1/(2*Sqrt[3]), -Sqrt[2/3], 0};
r[14] = {-1/2, 1/(2*Sqrt[3]), 1/Sqrt[6], -(1/Sqrt[2])};
r[15] = {0, 1/Sqrt[3], Sqrt[2/3], 0};
r[16] = {0, 1/Sqrt[3], -(1/Sqrt[6]), 1/Sqrt[2]};
r[17] = {0, -(1/Sqrt[3]), 1/Sqrt[6], 1/Sqrt[2]};
r[18] = {0, -(1/Sqrt[3]), -Sqrt[2/3], 0};
r[19] = {0, -(1/Sqrt[3]), 1/Sqrt[6], -(1/Sqrt[2])};
r[20] = {0, 1/Sqrt[3], -(1/Sqrt[6]), -(1/Sqrt[2])};
r[21] = {-1/2, 1/(2*Sqrt[3]), 1/Sqrt[6], 1/Sqrt[2]};
r[22] = {1/2, -1/(2*Sqrt[3]), -(1/Sqrt[6]), -(1/Sqrt[2])};
r[23] = {1/2, 1/(2*Sqrt[3]), 1/Sqrt[6], 1/Sqrt[2]};
r[24] = {-1/2, -1/(2*Sqrt[3]), -(1/Sqrt[6]), -(1/Sqrt[2])};
```

Rotations in 8 dimensions are special because of this triality symmetry.