Calculating the Klein-Gordon propagator

In this note I give the derivation of the Klein-Gordon propagator of the massive scalar field in spacetime of dimension d and Lorentz signature. In contrast to the previsous note where a derivation from the first-quantized theory was given, this note assumes the second-quantized field theory with bare action

$$S[\phi] = \int d^d x \, \left(\frac{1}{2} (\partial\phi)^2 - \frac{m}{2} \phi^2 \right).$$

The Green's function

We deploy the path integral technique to obtain a differential equation on the Klein-Gordon propagator $D(x^\mu = a^\mu - b^\mu)$:

$$D(a-b) = \langle \phi(a)\phi(b) \rangle = \int D\phi \, e^{iS[\phi]} \, \phi(a)\phi(b).$$

For this, consider an auxiliary functional $\Omega[\phi]$:

$$\Omega[\phi] = \phi(a).$$

We deploy the Schwinger "quantum action" principle for $\Omega[\phi]$:

$$0 = \delta \langle \Omega \rangle = \delta \int D\phi \, e^{iS[\phi]} \, \Omega[\phi] = \int D\phi \, e^{iS[\phi]} \left(\delta\Omega[\phi] + i\Omega[\phi] \, \delta S[\phi] \right) = \langle \delta\Omega \rangle + i \left\langle \Omega \, \delta S \right\rangle.$$

In our case,

$$\delta\Omega = \delta\phi(a) = \int d^d x \, \delta^{(d)}(x-a) \, \delta\phi(x),$$

$$\delta S = -\int d^d x \, \left(\Box + m^2 \right) \delta\phi(x).$$

Combining these equations, we get:

$$\left(\Box_x + m^2 \right) \langle \phi(a)\phi(x) \rangle = i \, \delta^{(d)}(x-a),$$

which is the desired differential equation on the propagator:

$$\left(\Box + m^2 \right) D(x) = i \, \delta^{(d)}(x).$$

We conclude that the propagator for the free scalar massive quantum field is equal to the Green's function of the differential operator $\Box + m^2$.

Fourier image of the propagator

The purpose of the present note is to solve for $D(x)$. We start by calculating its Fourier image,

$$\tilde{D}(p) = \int d^d x \, e^{-ip_\mu x^\mu} \, D(x).$$

We are interested in the Fourier image $\tilde{D}(p)$ because of the peculiar property of the Fourier transform: Fourier-transformed differential equations become algebraic, since

$$\partial_\mu \to ip_\mu.$$

Also, the Fourier image of the delta function is just 1. Hence we have an algebraic equation for the Fourier image $\tilde{D}(p)$:

$$\left(-p^2 + m^2 \right) \tilde{D}(p) = \left(-\omega^2 + \vec{p}^2 + m^2 \right) \tilde{D}(\omega, \vec{p}) = i,$$

$$\tilde{D}(p) = \frac{i}{-\omega^2 + \vec{p}^2 + m^2}.$$

The Wick rotation

Now we would like to calculate the propagator $D(x)$. It can be done via the inverse Fourier transform:

$$D(x) = \int \frac{d^d p}{(2\pi)^d} \, e^{ip_\mu x^\mu} \, \tilde{D}(p) = \int\limits_{-\infty}^{+\infty} d\omega \int d^{d-1}\vec{p} \, \frac{i\, e^{i\omega t} e^{-i\vec{p}\vec{x}} (2\pi)^{-d}}{-\omega^2 + \vec{p}^2 + m^2}.$$

The integral over ω is ill-defined as it stands, since it passes through the two poles:

$$\omega_\pm = \pm\sqrt{\vec{p}^2 + m^2}.$$

Hence, there are several different Green's functions. These correspond to the choice of the boundary conditions in the path integral. For quantum field theory, the Feynman propagator is usually chosen (since it coincides with the results obtained from canonical quantization): the poles are infinitesimally shifted along the complex plane, which is best described by the $i\varepsilon$ prescription:

$$D_F(x) = \int\limits_{-\infty}^{+\infty} d\omega \int d^{d-1}\vec{p} \, \frac{i\, e^{i\omega t} e^{-i\vec{p}\vec{x}} (2\pi)^{-d}}{-\omega^2 + \vec{p}^2 + m^2 - i\varepsilon}.$$

Now we would like to perform the Wick rotation of the integral above. We can smoothly deform the contour of integration in such a way that no pole transits through it. A convenient choice of the deformation of the integration contour is rotating it 90 degrees conterclockwise. Thus, ω becomes imaginary. It is convenient to pass to another (real) variable:

$$\omega = i\omega'.$$

The integral then becomes

$$D_F(x) = \int\limits_{-\infty}^{+\infty} i\, d\omega' \int d^{d-1}\vec{p} \, \frac{i\, e^{-\omega' t} e^{-i\vec{p}\vec{x}} (2\pi)^{-d}}{\omega'^2 + \vec{p}^2 + m^2}.$$

The $i\varepsilon$ prescription is no longer needed. It has served its purpose, which was to signal how the integration contour should be deformed.

The proper time method

In order to compute the integral above one could make use of the proper time method. We start with the trivial integral

$$\int\limits_0^\infty d\tau \, e^{-a\tau} = \frac{1}{a}.$$

This can be used to rewrite the expression for $D_F(x)$ as follows:

$$D_F(x) = -\int\limits_0^\infty d\tau \int\limits_{-\infty}^{+\infty} d\omega \int d^{d-1}\vec{p} e^{-\omega t} e^{-i\vec{p}\vec{x}} (2\pi)^{-d} \exp\left[-\tau\omega^2 - \tau\vec{p}^2 - \tau m^2\right].$$

Now the momenta integrals are Gaussian and can be evaluated:

$$D_F(x) = -\int\limits_0^\infty d\tau \, \frac{\pi^{d/2}}{(2\pi\sqrt{\tau})^d} \exp\left[\frac{1}{4\tau}\left(t^2 - \vec{x}^2\right) - \tau m^2\right] =$$

$$= -\frac{1}{(2\pi)^{d/2}} \left(\frac{\sqrt{m}}{-s^2}\right)^{\frac{d-2}{2}} \cdot K_{\frac{d}{2}-1}\left[m\sqrt{-s^2}\right].$$

For $d = 4$,

$$D_{F,d=4}(s) = \frac{-m}{4\pi^2\sqrt{-s^2}} \cdot K_1\left[m\sqrt{-s^2}\right].$$

Here we have introduced the interval

$$s^2 = t^2 - \vec{x}^2.$$

In order to handle the time-like intervals we can substitute $s \to is$:

$$D_{F,d=4}(s) = \begin{cases} \frac{-m}{4\pi^2\sqrt{-s^2}} \cdot K_1\left[m\sqrt{-s^2}\right], & s^2 < 0 \\ \frac{-im}{8\pi\sqrt{s^2}} \cdot H_1^{(2)}\left(m\sqrt{s^2}\right), & s^2 > 0. \end{cases}$$

Here we have used the property of the modified Bessel function:

$$K_\alpha(x) = \frac{\pi}{2}(-i)^{\alpha+1} H_\alpha^{(2)}(-ix), \quad \frac{\pi}{2} < \arg x \le \pi.$$

// TODO: the delta function $\delta(s)$ in the propagator.