# Interaction picture

In quantum mechanics, time evolution can be treated in several equivalent ways:

**Schrodinger picture:** quantum states are time-dependent; their evolution is governed by the *Schrodinger equation*:

$$i\frac{d}{dt}\left|\Psi(t)\right\rangle = \hat{H}\left|\Psi(t)\right\rangle.$$

Its solution can be formally expressed in terms of an operator exponential:

$$\left|\Psi(t)\right\rangle = e^{-i\hat{H}(t-t_0)}\left|\Psi(t_0)\right\rangle.$$

Quantum operators do not evolve with time: $\hat{a}(t) = \hat{a}(t_0)$. This renders the following evolution equation for observable matrix elements:

$$a_{\Phi\Psi}(t) = \left\langle\Phi(t)\right|\hat{a}(t)\left|\Psi(t)\right\rangle = \left\langle\Phi(t_0)\right|e^{i\hat{H}(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}(t-t_0)}\left|\Psi(t_0)\right\rangle.$$

It can be argued that quantum states contain complete information about the system as a whole, and therefore exist beyond space and time. The very question of whether states are objective (*psi-ontic* point of view) or subjective (*psi-epistemic* point of view) elements of reality is not currently settled.

Thus the idea of them evolving with time could be seen as counter-intuitive. While this might be true, since the evolution operator $e^{-i\hat{H}(t-t_0)}$ is *unitary*, time translations correspond to unitary linear maps on the space of states and therefore are isomorphisms of the Hilbert space structure. It means that we are still able to talk about fundamental states of the system and even preach psi-epistemism.

**Heisenberg picture:** quantum states are universal and time-independent. Instead, quantum operators (corresponding to the same observable quantity) differ with time. Their evolution is governed by the *Heisenberg equation*, which is a special case of the correspondence between Poisson brackets and commutators:

$$i\frac{d}{dt}\hat{a}(t) = \left[\hat{a}(t);\hat{H}\right].$$

Its solution is also given in terms of operator exponentials:

$$\hat{a}(t) = e^{i\hat{H}(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}(t-t_0)}.$$

Inserting this into the definition of the observable matrix element, we arrive at the same formula as in the Schrodinger picture:

$$a_{\Phi\Psi}(t) = \left\langle\Phi(t)\right|\hat{a}(t)\left|\Psi(t)\right\rangle = \left\langle\Phi(t_0)\right|e^{i\hat{H}(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}(t-t_0)}\left|\Psi(t_0)\right\rangle.$$

Here, yet another formalism is outlined. It is based on splitting of the Hamiltonian in two parts:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}.$$

Such splitting is, of course, unphysical. But it has proven useful, especially in perturbative computations.

**Interaction picture:** quantum operators evolve according to the Heisenberg equation, but with $\hat{H}_0$ (instead of the total Hamiltonian) in the right-hand side:

$$i\frac{d}{dt}\hat{a}(t) = \left[\hat{a}(t);\hat{H}_0\right];$$

$$\hat{a}(t) = e^{i\hat{H}_0(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}_0(t-t_0)}.$$

Quantum states also evolve in time. The law, which governs this evolution, has to be chosen such that observable matrix elements evolve in the same manner as in other pictures. Such evolution is given by the unitary operator

$$\hat{U}(t,t_0) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)};$$

$$\left|\Psi(t)\right\rangle = \hat{U}(t,t_0)\left|\Psi(t_0)\right\rangle.$$

We could check the latter proposal by inserting the expression for $\hat{U}$ in the matrix element:

$$a_{\Phi\Psi}(t) = \left\langle\Phi(t)\right|\hat{a}(t)\left|\Psi(t)\right\rangle = \left\langle\Phi(t_0)\right|\hat{U}^\dagger(t,t_0)e^{i\hat{H}_0(t-t_0)}$$

$$\hat{a}(t_0)e^{-i\hat{H}_0(t-t_0)}\hat{U}(t,t_0)\left|\Psi(t_0)\right\rangle = \left\langle\Phi(t_0)\right| e^{i\hat{H}(t-t_0)}e^{-i\hat{H}_0(t-t_0)}e^{i\hat{H}_0(t-t_0)}$$

$$\hat{a}(t_0)e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-i\hat{H}(t-t_0)}\left|\Psi(t_0)\right\rangle = \left\langle\Phi(t_0)\right| e^{i\hat{H}(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}(t-t_0)}\left|\Psi(t_0)\right\rangle,$$

which is indeed exactly what we would get in any other picture.

# Relation to the Heisenberg picture

When working in interaction picture, we must also keep track of Heisenberg-picture operators and states, because they define the observable matrix elements we want to calculate. This is why an obuse of notation is used:

States and operators, subscripted by $I$, belong to the interaction picture:

$$\hat{a}_I(t),\ \left|\Psi_I(t)\right\rangle,\ ...$$

However, the same states and operators without the $I$ subscript belong to the Heisenberg picture. They (by definition) coincide with subscripted quantities at $t = t_0$. Thus $t_0$ plays a crucial role in defining the interaction picture.
Instead of writing $\hat{H}_{I\,int}$ for the interaction Hamiltonian in the interaction picture, we just write $\hat{H}_I$. This can provoke confusion since it looks like a total Hamiltonian in interaction picture, but it is a common practice, sadly.

We could try to express $\hat{a}(t)$ in terms of $\hat{a}_I(t)$:

$$\hat{a}(t) = e^{i\hat{H}(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}(t-t_0)};$$

$$\hat{a}_I(t) = e^{i\hat{H}_0(t-t_0)}\hat{a}(t_0)e^{-i\hat{H}_0(t-t_0)};$$

$$\hat{a}(t_0) = e^{-i\hat{H}_0(t-t_0)}\hat{a}_I(t)e^{i\hat{H}_0(t-t_0)};$$

$$\hat{a}(t) = e^{i\hat{H}(t-t_0)}e^{-i\hat{H}_0(t-t_0)}\hat{a}_I(t)e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)} = \hat{U}^\dagger(t,t_0)\hat{a}_I(t)\hat{U}(t,t_0).$$

Also, because states are time-independent in Heisenberg picture,

$$\left|\Psi(t)\right\rangle = \left|\Psi(t_0)\right\rangle = \left|\Psi_I(t_0)\right\rangle;$$

$$\left|\Psi_I(t)\right\rangle = \hat{U}(t,t_0)\left|\Psi_I(t_0)\right\rangle = \hat{U}(t,t_0)\left|\Psi(t)\right\rangle.$$

States and operators from Heisenberg- and interaction- pictures are related through

$$\hat{a}(t) = \hat{U}^\dagger(t,t_0)\,\hat{a}_I(t)\,\hat{U}(t,t_0);$$

$$\left|\Psi\right\rangle = \hat{U}^\dagger(t,t_0)\left|\Psi_I(t)\right\rangle.$$

# The Dyson formula

Now we derive the Dyson formula for $\hat{U}(t,t_0)$. We start by differentiating the definition of $\hat{U}$ with respect to $t$:

$$\hat{U}(t,t_0) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)};$$

$$i\frac{d}{dt}\hat{U}(t,t_0) = -e^{i\hat{H}_0(t-t_0)}\,\hat{H}_0\,e^{-i\hat{H}(t-t_0)} + e^{i\hat{H}_0(t-t_0)}\,\hat{H}\,e^{-i\hat{H}(t-t_0)} =$$

$$e^{i\hat{H}_0(t-t_0)}\hat{H}_{int}e^{-i\hat{H}(t-t_0)} = e^{i\hat{H}_0(t-t_0)}\hat{H}_{int}e^{-i\hat{H}_0(t-t_0)}e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}$$

$$= \hat{H}_I(t)\hat{U}(t,t_0).$$

We are therefore solving a Schrodinger-like equation

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_I(t)\,\hat{U}(t,t_0)$$

with an initial condition

$$\hat{U}(t_0, t_0) = 1.$$

Unlike the usual Schrodinger equation in the Schrodinger picture, it can not be solved by exponentiation, because in general $\left[\hat{U}(t,t_0); \hat{H}_I(t)\right] \neq 0$. The differential equation for $\hat{U}(t,t_0)$ can be rewritten in a, perhaps, more familiar to the reader form:

$$iD \circ \hat{U}(t,t_0) = 0; \quad D = \frac{d}{dt} + i\hat{H}_I(t).$$

This exactly resembles the parallel-transport equations from gauge/Riemannian geometry. The "covariant derivative" $D$ contains the "gauge connection" $\hat{H}_I(t)$ which does not commute with the "holonomy" operator $\hat{U}(t,t_0)$.

Therefore, we can use the general solution for the "holonomy" of the "gauge connection" along the $[t_0, t]$ interval:

$$\hat{U}(t,t_0) = \mathrm{T}\ \exp\left\{-i\int_{t_0}^{t} dt'\,\hat{H}_I(t')\right\},$$

where $T\exp$ is the time-ordered exponential, which is rather formal an should be understood as a Tailor series with each term time-ordered. This formula is called *the Dyson formula*.

Let's prove that the time-ordered exponential solves exactly the differential equation for $\hat{U}(t,t_0)$ above:

$$\hat{U}(t,t_0) = \mathrm{T}\ \exp\left\{-i\int_{t_0}^{t} dt'\,\hat{H}_I(t')\right\} =$$

$$1 + (-i)\int_{t_0}^{t} dt'\,\hat{H}_I(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\,\hat{H}_I(t')\hat{H}_I(t'') + \ldots;$$

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{H}_I(t) + (-i)\int_{t_0}^{t} dt'\,\hat{H}_I(t)\hat{H}_I(t') + \ldots = \hat{H}_I(t)\,\hat{U}(t,t_0).$$

So the Dyson formula indeed solves exactly the differential equation for $\hat{U}(t,t_0)$.

The **Dyson formula** relates the interaction-picture evolution operator $\hat{U}(t,t_0)$ to the interaction-picture interaction Hamiltonian $\hat{H}_I(t)$:

$$\hat{U}(t,t_0) = \mathrm{T}\ \exp\left\{-i\int_{t_0}^{t} dt'\,\hat{H}_I(t')\right\}.$$

## Unitary evolution of states

Finally, let's define $\hat{U}(t_a, t_b)$ in such a way that

- It coincides with our definition of $\hat{U}(t,t_0) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}$,
- The composition law $\hat{U}(t_a,t_b)\,\hat{U}(t_b,t_c) = \hat{U}(t_a,t_c)$ is obeyed.

The obvious definition is

$$\hat{U}(t_a,t_b) = \hat{U}(t_a,t_0)\,\hat{U}^\dagger(t_b,t_0) = e^{i\hat{H}_0(t_a-t_0)}e^{-i\hat{H}(t_a-t_b)}e^{-i\hat{H}_0(t_b-t_0)}.$$

It obviously reduces to the definition of $\hat{U}(t, t_0)$ when $t_b = t_0$. Lets check the composition law:

$$\hat{U}(t_a, t_b)\,\hat{U}(t_b, t_c) = e^{i\hat{H}_0(t_a - t_0)} e^{-i\hat{H}(t_a - t_b)} e^{-i\hat{H}_0(t_b - t_0)} e^{i\hat{H}_0(t_b - t_0)}$$

$$e^{-i\hat{H}(t_b - t_c)} e^{-i\hat{H}_0(t_c - t_0)} = e^{i\hat{H}_0(t_a - t_0)} e^{-i\hat{H}(t_a - t_c)} e^{-i\hat{H}_0(t_c - t_0)} = \hat{U}(t_a, t_c).$$

The evolution of states in interaction picture between two arbitrary moments of time is determined by the unitary evolution operator

$$\hat{U}(t_a, t_b) = e^{i\hat{H}_0(t_a - t_0)} e^{-i\hat{H}(t_a - t_b)} e^{-i\hat{H}_0(t_b - t_0)} = \mathrm{T}\ \exp\left\{ -i \int_{t_a}^{t_b} dt\, \hat{H}_I(t) \right\}.$$

The composition law $\hat{U}(t_a, t_b)\,\hat{U}(t_b, t_c) = \hat{U}(t_a, t_c)$ is satisfied. In particular, it means that

$$\hat{U}^{-1}(t_a, t_b) = \hat{U}^{\dagger}(t_a, t_b) = \hat{U}(t_b, t_a).$$

The moment of time $t = t_0$ is given a special treat in the interaction picture formalism, since at $t = t_0$ all the interaction-picture objects exactly resemble the same objects in Heisenberg picture. Therefore, the existence of special $t_0$ must be understood as necessary condition for matching values between two pictures. The numeric value of $t_0$ has no physical meaning: observable matrix elements are $t_0$ -independent.