Hypersurface deformation algebra

In this note we follow the investigation of generic properties of Nambu-Goto bosonic p-branes (extended spacetime objects, generalizations of particles and strings) given by T. Thiemann from Perimeter Institute for Theoretical Physics.

Consider a Nambu-Goto p-brane given by an embedding of worldvolume manifold M into the (pseudo-)Riemannian target manifold T with metric $\eta_{\mu\nu}$. The physical properties of the embedding can be encoded in the Nambu-Goto action:

$$S[X] = -\int\limits_M d^p y \sqrt{-\det\left(q_{\alpha\beta}\right)},$$

where

$$q_{\alpha\beta}(y) \equiv \eta_{\mu\nu}\left(X(y)\right) \cdot \frac{\partial X^\mu(y)}{\partial y^\alpha} \cdot \frac{\partial X^\nu(y)}{\partial y^\beta}.$$

Canonical analysis

Suppose that $M = \mathbb{R} \times \Sigma$, where Σ is a fixed yet arbitrary $(p-1)$-dimensional manifold. We then choose the canonical coordinates on M:

$$y^\alpha = \{t, y^a\}, \quad \alpha \in 0 \dots p-1, \quad a \in 1 \dots p-1.$$

The canonically conjugate to $X^\mu(y)$ momenta are given by differentiation of the Nambu-Goto action:

$$\pi_\mu(t, y^a) = \frac{\delta S[X]}{\delta\left(\partial_t X(t, y^a)\right)} = -\sqrt{-\det\left(q_{\alpha\beta}\right)} \cdot q^{t\alpha} \cdot \eta_{\mu\nu}\left(X(t, y^a)\right) \cdot \partial_\alpha X^\mu(t, y^a).$$

The set of coordinates $\{X^\mu(y^a), \pi_\mu(y^a)\}$ labeled by a point on Σ completely parameterize the extended phase space. The Poisson algebra reads:

$$\{X^\mu(y^a);\ X^\nu(z^a)\} = 0,$$
$$\{\pi_\mu(y^a);\ \pi_\nu(z^a)\} = 0,$$
$$\{\pi_\mu(y^a);\ X^\nu(z^a)\} = \delta^\nu_\mu \cdot \delta\left(y, z\right).$$

Algebra of constraints

The diffeomorphism group algebra vect(M) acts on the phase space by means of p constraints at each point of Σ:

$$D_a = \pi_\mu \cdot \partial_a X^\mu,$$

$$C = \frac{1}{2}\left(\eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q_{ab})\right).$$

Lets compute the Poisson algebra of these constraints.

$$\{D_a(y);\ D_b(z)\} = \{\pi_\mu(y) \cdot \partial_a X^\mu(y);\ \pi_\nu(z) \cdot \partial_b X^\nu(z)\} =$$
$$= \left(\partial_a \pi_\mu(y) \cdot \partial_b X^\mu(z) - \partial_a X^\mu(y) \cdot \partial_b \pi_\mu(z)\right) \cdot \delta(y, z);$$

$$\{D_a(y);\ C(z)\} = \frac{1}{2}\{\pi_\sigma \cdot \partial_a X^\sigma;\ \eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q_{ab})\} =$$
$$= \left(\eta^{\mu\nu} \cdot \partial_a \pi_\mu(y) \cdot \pi_\nu(z) - \partial_a X^\mu(y) \cdot \partial^2_{bc} X^\nu(z) \cdot q^{bc}(z) \cdot \det q(z)\right)\delta(y, z);$$

$$\{C(y);\ C(z)\} = \left(\pi_\mu(z) \cdot \partial^2_{ab} X^\mu(y) \cdot q^{ab}(y) \cdot \det q(y) - \pi_\mu(y) \cdot \partial^2_{ab} X^\mu(z) \cdot q^{ab}(z) \cdot \det q(z)\right)\delta(y, z).$$

Smeared constraints

We introduce a set of smeared constraints defined on scalar and vector functions on Σ:

$$D(\vec{N}(y^a)) = \int_\Sigma d^{p-1}y \cdot N^a(y) \cdot D_a(y),$$

$$C(N(y^a)) = \int_\Sigma d^{p-1}y \cdot N(y) C(y).$$

We now wish to compute the Poisson algebra of the smeared constraints:

$$\left\{D(\vec{N});\ D(\vec{M})\right\} = \int_\Sigma d^{p-1}y \int_\Sigma d^{p-1}z\, N^a(y)\, M^b(z)\, \{D_a(y);\ D_b(z)\} =$$

$$= \int_\Sigma d^{p-1}y \int_\Sigma d^{p-1}z\, N^a(y)\, M^b(z)\, (\partial_a \pi_\mu(y) \cdot \partial_b X^\mu(z) - \partial_a X^\mu(y) \cdot \partial_b \pi_\mu(z))\, \delta(y,z) =$$

$$= \int_\Sigma d^{p-1}y \int_\Sigma d^{p-1}z \left(-\partial_a N^a(y) \cdot M^b(z) \cdot \pi_\mu(y) \cdot \partial_b X^\mu(z) + N^a(y) \cdot \partial_b M^b(z) \cdot \partial_a X^\mu(y) \cdot \pi_\mu(z)\right) \delta(y,z) =$$

$$= \alpha' \int_\Sigma d^{p-1}y \left(N^a \cdot \partial_b M^b \cdot \partial_a X^\mu \cdot \pi_\mu - \partial_a N^a \cdot M^b \cdot \pi_\mu \cdot \partial_b X^\mu\right) =$$

$$= \alpha' \int_\Sigma d^{p-1}y \left(N^a \partial_b M^b - M^a \partial_b N^b\right) \cdot \pi_\mu \partial_a X^\mu = D\left(\mathcal{L}_{\vec{N}} \vec{M}\right);$$

$$\left\{D(\vec{N});\ C(F)\right\} = \int_\Sigma d^{p-1}y \int_\Sigma d^{p-1}z\, (N^a(y) F(z) \cdot \partial_a \pi_\mu(y) \cdot \pi_\nu(z) \cdot \eta^{\mu\nu} -$$

$$- N^a(y) F(z) \cdot \partial_a X^\mu(y) \cdot \partial^2_{bc} X^\nu(z) \cdot q^{bc}(z) \cdot \det q(z)\big)\, \delta(y,z) =$$

$$= \int_\Sigma d^{p-1}y\, (-\partial_a N^a \cdot F)\, (\eta^{\mu\nu} \pi_\mu \pi_\nu + \det q(z)) = C(\mathcal{L}_{\vec{N}} F);$$

$$\{C(F);\ C(G)\} = \int_\Sigma d^{p-1}y \int_\Sigma d^{p-1}z \left(F(y) G(z) \cdot \pi_\mu(z) \cdot \partial^2_{ab} X^\mu(y) \cdot q^{ab}(y) \cdot \det q(y) -\right.$$

$$\left. - F(y) G(z) \cdot \pi_\mu(y) \cdot \partial^2_{ab} X^\mu(z) \cdot q^{ab}(z) \cdot \det q(z)\right) \delta(y,z) =$$

$$= \int_\Sigma d^{p-1}y\, (\partial_a F(y) \cdot G(y) - \partial_a G(y) \cdot F(y)) \det q(y) \cdot q^{ab}(y) \cdot D_b(y).$$

Deformation algebra

We are left with the following algebraic relations:

$$\left\{D(\vec{N});\ D(\vec{M})\right\} = D\left(\mathcal{L}_{\vec{N}} \vec{M}\right),$$

$$\left\{D(\vec{N});\ C(F)\right\} = C\left(\mathcal{L}_{\vec{N}} F\right),$$

$$\{C(F);\ C(G)\} = \int_\Sigma d^{p-1}y\, (\partial_a F(y) \cdot G(y) - F(y) \cdot \partial_a G(y)) \cdot \det(q_{ab}(y)) \cdot q^{ab}(y) \cdot D_b(y).$$

For $p > 2$ we get nontrivial structure constants depending on q_{ab}. This also happens in quantum General Relativity.

For $p = 2$ (the string) $\dim q_{ab} = 1 \times 1$ and $\det(q_{ab}(y)) \cdot q^{ab}(y) = 1$. The deformation algebra is then isomorphic to the Lie algebra of the string worldsheet diffeomorphism group.