Variation of the holonomy

In this note I compute the first variation of the holonomy functional $h_\gamma[A]$ with respect to the component of the gauge connection $A^a_\mu(x)$.

Holonomy functional

Consider a smooth n-dimensional manifold M. The holonomy functional h_γ associated to the smooth curve $\gamma:\ [0,1]\to M$ associates an element $h_\gamma[A]$ of the gauge group G to any gauge connection $A(x)$. This group element is defined to be the unique solution of the matrix-valued differential equation

$$\frac{d}{d\tau}h^{(A)}_\gamma(\tau) = -a(\tau)\cdot h^{(A)}_\gamma(\tau)$$

taken at $\tau = 1$:

$$h_\gamma[A] = h^{(A)}_\gamma(\tau=1),$$

where

$$a(\tau) = A_\mu(\gamma(\tau))\cdot\dot{\gamma}^\mu(\tau).$$

Claim #1: Holonomy is independent of parametrization

In order to define the holonomy as associated to the *curve* γ we have to prove that the defining differential equation is reparametrization-independent. Indeed, choose an arbitrary reparametrization $\sigma(\tau)$ such that

$$\begin{cases}\sigma(\tau=0)=0,\\ \sigma(\tau=1)=1.\end{cases}$$

Consider the new differential equation for the holonomy:

$$\frac{d}{d\sigma}h^{(A)}_\gamma(\sigma) = -a(\sigma)\cdot h^{(A)}_\gamma(\sigma),$$

$$\frac{d\tau}{d\sigma}\cdot\frac{d}{d\tau}h^{(A)}_\gamma(\tau) = -a(\sigma)\cdot h^{(A)}_\gamma(\tau).$$

This equation holds automatically provided that $a(\tau)$ transforms as an 1-form:

$$a(\sigma)\cdot d\sigma = a(\tau)\cdot d\tau,$$

which is exactly what we would get by considering the transformation properties of the gauge connection $A(x)$, since $a(\tau)$ is the connection induced by A on the curve γ. Thus the defining differential equation for the holonomy is reparametrization-invariant Q.E.D.

Claim #2: Holonomy transforms as a two-point tensor

Consider an arbitrary gauge transformation $\omega(x)$. By definition, the gauge connection transforms according to

$$A\to\omega A\omega^{-1}+\omega d\omega^{-1}.$$

We claim that the holonomy transforms according to

$$h_\gamma\to\omega(t)h_\gamma\omega^{-1}(s),$$

where $s=\gamma(\tau=0)$ and $t=\gamma(\tau=1)$ are the two endpoints of the curve γ. In order to prove this, consider the defining equation for the holonomy:

$$h^{(A)}_\gamma(\tau)\to\omega(\gamma(\tau))h^{(A)}_\gamma(\tau)\omega^{-1}(s),$$

$$\frac{d}{d\tau}\omega(\gamma(\tau))h^{(A)}_\gamma(\tau)\omega^{-1}(s) = d\omega(\dot{\gamma}(\tau))\cdot h^{(A)}_\gamma(\tau)\omega^{-1}(s)+\omega(\gamma(\tau))\cdot\frac{d}{d\tau}h^{(A)}_\gamma(\tau)\cdot\omega^{-1}(s) =$$

$$= d\omega(\dot{\gamma}(\tau))\cdot h^{(A)}_\gamma(\tau)\omega^{-1}(s)-\omega(\gamma(\tau))a(\tau)h^{(A)}_\gamma(\tau)\omega^{-1}(s) = -\left(\omega A\omega^{-1}+\omega d\omega^{-1}\right)(\dot{\gamma})\cdot\omega(\gamma(\tau))h^{(A)}_\gamma(\tau)\omega^{-1}(s).$$

We obtain the defining equation for the transformed holonomy in terms of the transformed connection, thus proving the correctness of the proposed transformation law.

Claim #3: Holonomy is given by the path-ordered exponential

Lets prove the following formula:

$$h_\gamma[A] = \mathrm{P}\exp\left\{-\int_\gamma a\right\} = \mathrm{P}\exp\left\{-\int_0^1 d\tau\, A_\mu(\gamma(\tau))\,\dot\gamma^\mu(\tau)\right\},$$

where the path-ordered exponential (P exp) operator is defined through its Taylor series:

$$h_\gamma[A] = \sum_{k=0}^{\infty}\left\{\frac{1}{k!}\prod_{i=1}^{k}\int_0^1 d\tau_i \times \mathrm{P}\left(a(\tau_1)\dots a(\tau_k)\right)\right\} = \sum_{k=0}^{\infty}\left\{\prod_{i=1}^{k}\int_0^{\tau_{i-1}} d\tau_i \times a(\tau_1)\dots a(\tau_k)\right\}.$$

Consider the k-th order in a term in this expansion of $h_\gamma^{(A)}(\tau)$:

$$h_{\gamma,k}^{(A)}(\tau) = \int_0^\tau d\tau_1 \times \prod_{i=2}^{k}\int_0^{\tau_{i-1}} d\tau_i \times a(\tau_1)\dots a(\tau_k).$$

We calculate its derivative with respect to τ. Only the $d\tau_1$ integral contributes, giving

$$\frac{d}{d\tau}h_{\gamma,k}^{(A)}(\tau) = \int_0^\tau d\tau_2 \times \prod_{i=3}^{k}\int_0^{\tau_{i-1}} d\tau_i \times a(\tau)a(\tau_2)\dots a(\tau_k) = a(\tau)h_{\gamma,k-1}^{(A)}(\tau).$$

But this is exactly the k-th order in a term in the expansion of

$$a(\tau)h_\gamma^{(A)}(\tau).$$

Thus the defining equation is satisfied by the path-ordered exponential. Since the Cauchy problem can not have multiple solutions, we conclude that the path-ordered exponential gives the only solution of the defining equation for the holonomy, and thus is equal to the holonomy, Q.E.D.

Variation of the holonomy

We prove the following formula:

$$\frac{\delta h_\gamma[A]}{\delta A_\mu^a(x)} = \int_0^1 d\tau\, \delta^{(n)}(x;\gamma(\tau))\,\dot\gamma^\mu(\tau)\cdot h_{\gamma_2}[A]\,t_a\,h_{\gamma_1}[A].$$

Here we have adopted the following expression for the components of the gauge connection:

$$A(x) = A_\mu^a(x)\,t_a\,dx^\mu.$$

The curves γ_1 and γ_2 are equal to the two portions of the curve γ separated by the point $\gamma(\tau)$, and are thus dependent on $\tau \in [0,1]$. The distributional delta function $\delta^{(n)}(x;\gamma(\tau))$ which arises in this formula is due to notation and is annihilated by the $d^n x$ integration when considering the variational differential $\delta h_\gamma[A]$.

Consider the k-th order in a term in the expansion for the path-ordered exponential:

$$h_{\gamma,k}[A] = \prod_{i=1}^{k}\int_0^{\tau_{i-1}} d\tau_i \times a(\tau_1)\dots a(\tau_k)$$

and consider its variation with respect to $A_\mu^a(x)$:

$$\frac{\delta a(\tau)}{\delta A_\mu^a(x)} = \frac{\delta\left(A_\nu^b(\gamma(\tau))\cdot\dot\gamma^\nu\cdot t_b\right)}{\delta A_\mu^a(x)} = \delta^{(n)}(x;\gamma(\tau))\cdot\dot\gamma^\mu(\tau)\cdot t_a,$$

$$\frac{\delta h_{\gamma,k}[A]}{\delta A^a_\mu(x)} = \sum_{u,v} \left[\prod_{i=1}^{u} \int_0^{\tau_{i-1}} d\tau_i \times a(\tau_1) \dots a(\tau_u) \right] \times$$

$$\times \int d\tau\, \delta^{(n)}(x;\gamma(\tau)) \cdot \dot{\gamma}^\mu(\tau) \cdot t_a \times \left[\prod_{j=1}^{v} \int_\tau^{\tau_{j-1}} d\tau_j \times a(\tau_1) \dots a(\tau_v) \right],$$

where the sum is over naturals satisfying $u + v = k - 1$ and the first u integrals can go only to τ (not all the way to 1), while the last v integrals start from τ (not all the way from 0). The sum comes from the fact that each integral in the original expression contributes to the variation.

Now it is obvious that when expanding the two holonomies h_{γ_1} and h_{γ_2} in the variational formula above we will get the same sum over all u and v, which is equivalent to summing over all u, v satisfying $u + v = k - 1$ and then over all possible values of k. Thus the variational formula is proven.