Hamiltonian analysis of Holst action

Consider a generalization of the Palatini action for classical General Relativity with a topological term — the *Holst action*:

$$S[\underline{e},\underline{\omega}] = \frac{1}{16\pi G}\int \underline{e}^I \wedge \underline{e}^J \wedge \left(\underline{\underline{*F}}_{IJ} + \frac{1}{\gamma}\underline{\underline{F}}_{IJ} \right) = \frac{1}{16\pi G}\int e\cdot e^\alpha_I e^\beta_J \left(F_{\alpha\beta}{}^{IJ} - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}F_{\alpha\beta}{}^{KL} \right),$$

where the Cartan structure equations define the torsion and curvature as

$$\underline{\underline{T}}^I = \underline{\underline{\nabla e}}^I = \underline{\underline{de}}^I + \underline{\omega}^I{}_J \wedge \underline{e}^J,$$

$$\underline{\underline{F}}^I{}_J = \underline{\underline{d\omega}}^I{}_J + \underline{\omega}^I{}_K \wedge \underline{\omega}^K{}_J,$$

and the Hodge star acts on flat indices as

$$\underline{\underline{*F}}_{IJ} = \frac{1}{2}\epsilon_{IJKL}\underline{\underline{F}}^{KL}.$$

Canonical variables

After the spacetime foliation, we proceed with an unusual choice of canonical variables — the *Ashtekar-Barbero connection*:

$$\begin{cases} A^i_a = \frac{1}{2}\epsilon^i{}_{jk}\omega_a{}^{jk} + \gamma\omega_a{}^{0i}, \\ E^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon_{abc}e^j_b e^k_c. \end{cases}$$

Note that while A^i_a is a true 3-dimensional vector, E^a_i is actualy a vector density, since it contains the densitized Levi-Civita symbol ϵ_{abc} in its definition.

Also, A^i_a does not have nice $SO(3,1)$ transformation properties, unless a specific value is given to the *Barbero-Immirzi constant*: $\gamma = i$, in which case we recover the usual *Ashtekar connection* which transforms in the self-dual representation of $SO(3,1)$. But unless γ is real, the Ashtekar-Barbero connection becomes complex, and nontrivial reality conditions have to be imposed afterwards.

Nevertheless, objects A^i_a and E^a_i are well-defined in all gauges and coordinate systems, despite having nonlinearities in their transformation properties.

The vector density E^a_i has a nice geometrical interpretation. Consider a surface embedded into space: $x^a = \chi^a(\sigma^1, \sigma^2)$. The area of the surface is given by an integral

$$\mathcal{A} = \int d^2\sigma\,\sqrt{\det f}$$

with the induced metric given by

$$f_{xy} = q_{ab}(\chi)\cdot\frac{\partial\chi^a}{\partial\sigma^x}\frac{\partial\chi^b}{\partial\sigma^y} = \delta_{ij}\,e^i_a(\chi)\,e^j_b(\chi)\,\frac{\partial\chi^a}{\partial\sigma^x}\frac{\partial\chi^b}{\partial\sigma^y},$$

It follows that

$$\mathcal{A} = \int d^2\sigma\,\sqrt{E^a_i n_a E^b_i n_b},$$

where

$$n_a(\sigma) = \epsilon_{abc}\frac{\partial\chi^b}{\partial\sigma^1}\frac{\partial\chi^c}{\partial\sigma^2}.$$

Thus, E^a_i contains information about areas of surfaces, and indeed, after appropriate regularization, from this classical quantity the quantum area operator of LQG is built.

Hamiltonian analysis

Now we have to rewrite the Holst action in terms of the Ashtekar-Barbero connection.

$$F^{IJ}{}_{\mu\nu} = \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \eta_{KL}\,\omega_{[\mu}{}^{IK}\omega_{\nu]}{}^{LJ}$$