Group of loops

A *curve* in a differentiable manifold M is an equivalence class of maps such as

$$p:\ [0..s_1]\cap\cdots\cap[s_{n-1}..1]\to M,$$

smooth in each interval $[s_i..s_{i+1}]$ and continuous in the whole domain $[0..1]$, under reparametrizations.

A *concatenation* of curves is defined for any two representatives as

$$[p_1\circ p_2]\,(s)=\begin{cases}p_1(2s) & s\in[0..1/2],\\ p_2(2s-1) & s\in[1/2..1].\end{cases}$$

A curve can be *traversed*, which changes its orientations to the opposite:

$$p^{-1}(s)=p(1-s).$$

As mentioned above, curves are equivalence classes of functions (parameterized curves) under reparametrizations. So any two functions are considered equivalent ($p_1\sim p_2$) whenever there exists a diffeomorphism $\varphi:\ [0..1]\to[0..1]$, such that

$$p_1(s)=p_2(\varphi(s)).$$

A set of closed curves starting and ending at point $o\in M$ is denoted as L_o. It forms a semigroup under concatenation:

- The semigroup product is just a concatenation of closed curves. Obviously it belongs to L_o since it starts and ends at o.
- The semigroup identity is the null curve $e(s)=0$.
- The inverse can be defined as traversed curve, but it does not satisfy the axioms from the definition of the group. In particular, $p\circ p^{-1}\neq e$. Thus, L_o is not a group, but a semigroup.

Could we somehow overcome this problem and make L_o a group? We have to identify objects like $p\circ p^{-1}$ and the identity curve.

A closed curve is called *thin* if it encloses no area. A comprehensive definition can be given in terms of subpaths: each constituent subpath of the thin curve enters it two times, once ordinary and once reversed.

A *loop* is an equivalence class of closed curves under the following equivalence relation:

$$p\sim q\quad\Longleftrightarrow\quad p\circ q^{-1}\text{ is thin.}$$

Since $p\circ p^{-1}$ is always thin, it is equivalent to the semigroup identity. Thus, the set of loops starting and ending at the point o forms a group of loops $\mathcal{L}_o$.