Geodesics in high-degree geometry

By geometry of degree n I mean a class of equivalence of symmetric covariant tensors of rank n under diffeomorphisms. The object in the equivalence class is called the metric of degree n:

$$g(x) = g_{\mu_1\ldots\mu_n}(x)\, dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_n}.$$

Under an infinitesimal diffeomorphism generated by the vector field $\xi(x)$ the metric transforms according to

$$g(x) \to g(x) + \mathcal{L}_\xi g(x) \cdot \delta\varepsilon,$$

$$g_{\mu_1\ldots\mu_n} \to g_{\mu_1\ldots\mu_n} + g_{\mu_1\ldots\mu_n}\partial_\nu \xi^\nu + \sum_{i=1}^{n} \xi^\nu \partial_{\mu_i}\, g_{\mu_1\ldots\nu\ldots\mu_n},$$

where $\mathcal{L}$ is the Lie derivative.

Geodesics

The length of the path is given by the generalization of the Riemannian formula:

$$\mathrm{Len}[x] = \int d\tau\, \sqrt[n]{g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}}.$$

It is straightforward to show the reparametrization invariance of this expression. Its variation with respect to $x(\tau)$ gives the differential equation for the geodesic.

$$\delta\mathrm{Len}[x] = \int d\tau \left[\frac{\frac{\partial}{\partial x^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)}{n\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)^{1-\frac{1}{n}}} - \frac{d}{d\tau}\left(\frac{\frac{\partial}{\partial \dot{x}^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)}{n\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)^{1-\frac{1}{n}}} \right) \right] \cdot \delta x^\nu(\tau) = 0,$$

$$\frac{\frac{\partial}{\partial x^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)}{n\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)^{1-\frac{1}{n}}} - \frac{d}{d\tau}\left(\frac{\frac{\partial}{\partial \dot{x}^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)}{n\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right)^{1-\frac{1}{n}}} \right) = 0.$$

In the proper length gauge, τ is defined to measure the proper length along the curve and the constraint is given by

$$g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} = 0.$$

Thus, the geodesic equation in the proper length gauge reduces to

$$\frac{\partial}{\partial x^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right) - \frac{d}{d\tau}\frac{\partial}{\partial \dot{x}^\nu}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}\right) = 0,$$

$$g_{\mu_1\ldots\mu_n,\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} - \frac{d}{d\tau}\left(g_{\mu_1\ldots\mu_n}(x) \cdot \sum_i \dot{x}^{\mu_1} \ldots \delta^{\mu_i}_\nu \ldots \dot{x}^{\mu_n} \right) = 0,$$

$$g_{\mu_1\ldots\mu_n,\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} - \frac{d}{d\tau}\left(n g_{\mu_1\ldots\mu_{n-1}\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_{n-1}} \right) = 0,$$

$$g_{\mu_1\ldots\mu_n,\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} - n g_{\mu_1\ldots\mu_{n-1}\nu,\sigma}(x) \cdot \dot{x}^\sigma \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_{n-1}} -$$
$$n(n-1) g_{\mu_1\ldots\mu_{n-1}\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_{n-2}} \cdot \ddot{x}^{\mu_{n-1}} = 0,$$

$$g_{\mu_1\ldots\mu_{n-1}\nu}(x) \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_{n-2}} \cdot \ddot{x}^{\mu_{n-1}} + \Gamma_{\mu_1\ldots\mu_n|\nu} \cdot \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} = 0,$$

where we have introduced the Christoffel symbols of degree n:

$$\Gamma_{\mu_1\ldots\mu_n|\nu} = \frac{1}{n(n-1)}\left(\sum_i g_{\mu_1\ldots\nu\ldots\mu_n,\mu_i} - g_{\mu_1\ldots\mu_n,\nu} \right).$$

Special cases

Lets look at some special cases.

For $n = 1$, something special happens. The equation of motion is no longer second-order and the generalized Christoffel symbols are undefined. The direct variation of the arc length of order 1 gives

$$\delta \int g_\mu(x)\dot{x}^\mu d\tau = 0 \implies (g_{\mu,\nu} - g_{\nu,\mu})\,\dot{x}^\mu = 0.$$

There are no geodesics in 1-degree geometry. Indeed, the metric has d independent components, which are all just gauge (there are d independent gauge parameters).

For $n = 2$, the usual Riemannian geodesics are recovered:

$$g_{\alpha\nu}\ddot{x}^\alpha + \Gamma_{\alpha\beta|\nu}\dot{x}^\alpha\dot{x}^\beta = 0,$$

$$\Gamma_{\alpha\beta|\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right).$$

This case is special, since $\dot{x}$ does not show up in the lhs. Thus it has a unique solution for any given initial values of coordinates and velocities.

For $n = 3$, the geodesic equation becomes

$$g_{\alpha\beta\nu}\dot{x}^\alpha\ddot{x}^\beta + \Gamma_{\alpha\beta\gamma|\nu}\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma = 0,$$

$$\Gamma_{\alpha\beta\gamma|\nu} = \frac{1}{6}\left(g_{\nu\beta\gamma,\alpha} + g_{\alpha\nu\gamma,\beta} + g_{\alpha\beta\nu,\gamma} - g_{\alpha\beta\gamma,\nu}\right).$$

For $n = 4$, the geodesic equation becomes

$$g_{\alpha\beta\gamma\nu}\dot{x}^\alpha\dot{x}^\beta\ddot{x}^\gamma + \Gamma_{\alpha\beta\gamma\delta|\nu}\dot{x}^\alpha\dot{x}^\beta\dot{x}^\gamma\dot{x}^\delta = 0,$$

$$\Gamma_{\alpha\beta\gamma\delta|\nu} = \frac{1}{12}\left(g_{\nu\beta\gamma\delta,\alpha} + g_{\alpha\nu\gamma\delta,\beta} + g_{\alpha\beta\nu\delta,\gamma} + g_{\alpha\beta\gamma\nu,\delta} - g_{\alpha\beta\gamma\delta,\nu}\right).$$

The Cauchy problem

Given the Cauchy initial data $\{x, \dot{x}\}$, is there always a unique solution $x(\tau)$? This can be traced back to the requirement for the matrix

$$f_{\alpha\beta} = g_{\alpha\beta\mu_1\dots\mu_{n-2}} \cdot \dot{x}^{\mu_1} \dots x^{\mu_{n-2}}$$

to be invertible. Thus the requirement on the metric of degree n is that any of its projections on two distinct indices gives an invertible matrix.