Feynman diagrams

In perturbative quantum field theory, we focus on calculation of the *correlations* of quantum fields. These are defined as

$$\langle \phi_1(x_1)\ldots\phi_n(x_n)\rangle = \langle\Omega|\,\mathrm{T}\,\hat{\phi}_1(x_1)\ldots\hat{\phi}_n(x_n)\,|\Omega\rangle\,,$$

where $\hat{\phi}_i$ are some quantum field operators in Heisenberg picture, taken at space-time points x_i; $|\Omega\rangle$ is the vacuum (lowest-energy) eigenstate of the total Hamiltonian $\hat{H}$ and the *chronological ordering symbol* T stands for time-ordering. As it will become clear later, all observable quantities in perturbative QFT (scattering amplitudes) can be derived from the correlations. It is also worth noting that quantum fields *commute* inside the correlation brackets, since the corresponding operators are time-ordered after all. Fermionic quantum fields *anticommute* inside the correlation brackets, since time-ordering is defined for them with an additional minus sign.

Perturbative expansions

Our strategy would be to re-express the correlations in terms of the interaction picture of quantum mechanics. For that, suppose that we have a natural splitting of the total Hamiltonian in two parts:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int},$$

where $\hat{H}_0$ corresponds to the Hamiltonian of the free theory, and $\hat{H}_{int}$ is treated as a small perturbation over $\hat{H}_0$. This implies, among other things, that we can choose the Fock space of $\hat{H}_0$ to be habitable by the states of the interacting quantum field theory.

Actually, this statement is not just imprecise, but also completely wrong. States of interacting quantum fields and the Fock space states of the corresponding free theory belong to **unitary-inequivalent** representations, which is the main content of Haar's theorem. It is not to be underestimated, because it completely forbids any kind of interaction picture in perturbative QFT. However, this issue can be resolved if we restrict ourselves to calculating correlation functions instead of working with an actual space of states. The scatering operator is then defined via the LSZ formalism — on so-called asymptotic states, which resemble the Fock space of the free Hamiltonian.

Nevertheless, we switch to the interaction picture in order to obtain a formal expression for the correlations. First, we express all the Heisenberg-picture quantum field operators in the correlation through the corresponding interaction-picture operators:

$$\hat{\phi}_k(x_k) = \hat{U}^\dagger(x_k^0, t_0)\,\hat{\phi}_{I\,k}(x_k)\,\hat{U}(x_k^0, t_0) = \hat{U}(t_0, x_k^0)\,\hat{\phi}_{I\,k}(x_k)\,\hat{U}(x_k^0, t_0),$$

where x_k^0 is the time-coordinate of the k-th space-time point in the correlation.

Now we wish to relate the Heisenberg picture vacuum of the interacting theory (the *true vacuum*) $|\Omega\rangle$ to the free theory vacuum $|0\rangle$, which only makes sense when defined in the interaction picture. This would help us bring up the calculation of the correlations to the one in the free theory (since it would involve the free theory vacuum and interaction-picture operators which are governed by the free theory Hamiltonian).

Consider the exponentiation of the total Hamiltonian operator of the interacting theory: $e^{-i\hat{H}(T+t_0)}$.

Given a basis of eigenstates of $\hat{H}$, this operator can be expanded as

$$e^{-i\hat{H}(T+t_0)} = \sum_N e^{-iE_N(T+t_0)}\,|N\rangle\,\langle N|\,.$$

Now comes a dirty trick: for large enough T all the interactions would “die out” and what remains left is the vacuum mode:

$$\lim_{T\to\infty} e^{-i\hat{H}(T+t_0)} \sim e^{-iE_\Omega(T+t_0)}\,|\Omega\rangle\,\langle\Omega|\,.$$

We could try to make this argument a little more mathematically precise at the cost of losing physical intuition by setting $T\to\infty(1-i\epsilon)$. In this case only the lowest-energy mode (the vacuum state) of the Hamiltonian operator survives exponentiation since the exponential now contains a small real part.

It could also be understood as follows: in the large T limit all the oscillations in the exponential become much more rapid then the vacuum mode, which gives the leading behavior. The last explanation provides some physical insights, but lacks mathematical precision.

Anyways, we use this trick to act with the mentioned above exponential on the free theory vacuum state $|0\rangle$:

$$\lim_{T\to\infty} e^{-i\hat{H}(T+t_0)}\,|0\rangle \sim e^{-iE_\Omega(T+t_0)}\,\langle\Omega|0\rangle\,|\Omega\rangle\,.$$

On the other hand, we choose the free theory Hamiltonian $\hat{H}_0$ to be normal-ordered, which means that the free theory vacuum energy is zero:

$$\hat{H}_0\,|0\rangle = 0; \quad\Longrightarrow\quad e^{i\hat{H}_0(T+t_0)}\,|0\rangle = |0\rangle\,.$$

Therefore,

$$e^{-i\hat{H}(T+t_0)}\,|0\rangle = e^{-i\hat{H}(T+t_0)}e^{i\hat{H}_0(T+t_0)}\,|0\rangle = e^{i\hat{H}((-T)-t_0)}e^{-i\hat{H}_0((-T)-t_0)}\,|0\rangle = \hat{U}^\dagger(-T,t_0)\,|0\rangle = \hat{U}(t_0,-T)\,|0\rangle\,.$$

Combining these two results, we get:

$$|\Omega\rangle \sim \frac{\hat{U}(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(T+t_0)}\,\langle\Omega|0\rangle},$$

where we implicitly assume that $T\to\infty$. Unless the Hilbert product of the two vacuums is zero (which would mean that $\hat{H}$ can in no way be treated as a perturbation of $\hat{H}_0$), the denominator is just a numerical constant.

Similarly, by acting with the Hermission conjugate exponential (in order to still take the $T\to\infty$ limit in the end) on the bra vacuum, we obtain:

$$\langle\Omega| \sim \frac{\langle 0|\,\hat{U}(T,t_0)}{e^{-iE_\Omega(T-t_0)}\,\langle 0|\Omega\rangle}.$$

The normalization condition dictates:

$$\langle\Omega|\Omega\rangle = \frac{\langle 0|\,\hat{U}(T,t_0)\,\hat{U}(t_0,-T)\,|0\rangle}{e^{-iE_\Omega(T-t_0)}e^{-iE_\Omega(T+t_0)}\,\langle\Omega|0\rangle\,\langle 0|\Omega\rangle} = \frac{\langle 0|\,\hat{U}(T,-T)\,|0\rangle}{e^{-2iE_\Omega T}\cdot|\langle 0|\Omega\rangle|^2} = 1;$$

$$e^{-2iE_\Omega T}\cdot|\langle 0|\Omega\rangle|^2 = \langle 0|\,\hat{U}(T,-T)\,|0\rangle\,.$$

Now we are ready to derive the expression for the correlations in the interaction picture.

$$\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle = \langle\Omega|\,\mathrm{T}\,\hat{\phi}_1(x_1)\ldots\hat{\phi}_n(x_n)\,|\Omega\rangle =$$

$$= \frac{1}{e^{-2iE_\Omega T}\cdot|\langle 0|\Omega\rangle|^2}\cdot\langle 0|\,\hat{U}(T,t_0)\,\mathrm{T}\left\{\hat{U}(t_0,x_1^0)\,\hat{\phi}_{I\,1}(x_1)\,\hat{U}(x_1^0,t_0)\ldots\right\}\,\hat{U}(t_0,-T)\,|0\rangle\,.$$

We can glue together the evolution operators between interaction-picture field operators (inside the chronological ordering symbol) by using the composition law:

$$\ldots\hat{\phi}_{I\,k}(x_k)\,\hat{U}(x_k^0,t_0)\hat{U}(t_0,x_{k+1}^0)\,\hat{\phi}_{I\,k+1}(x_{k+1})\ldots = \ldots\hat{\phi}_{I\,k}(x_k)\,\hat{U}(x_k^0,x_{k+1}^0)\,\hat{\phi}_{I\,k+1}(x_{k+1})\ldots.$$

The next step would be to deal carefully with time-ordering. Each of the evolution operators is already time-ordered, because there is a time-ordered exponential in the Dyson formula. There is more: we can shuffle the operators inside the chronological ordering symbol any way we want (since they will get ordered chronologically after all). The last observation would be that since the only two evolution operators outside the chronological ordering symbol are (because of Dyson formula) also time-ordered, we could move them inside without changing anything. After all, all the evolution operators (reshuffled the way we want, inside the chronological ordering brackets) can be nicely glued together to give $\hat{U}(T,-T)$:

$$\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle = \frac{1}{N}\cdot\langle 0|\,\mathrm{T}\left\{\hat{U}(T,-T)\,\hat{\phi}_{I1}(x_1)\ldots\hat{\phi}_{In}(x_n)\right\}|0\rangle\,,$$

where $N = e^{-2iE_\Omega T}\cdot|\langle 0|\Omega\rangle|^2$ is the normalization factor which we know (it was derived above) is equal to

$$N = \langle 0|\,\hat{U}(T,-T)\,|0\rangle = \langle 0|\,\mathrm{T}\,\hat{U}(T,-T)\,|0\rangle\,.$$

Correlations of interacting quantum fields can be expressed formally in the interaction picture:

$$\langle\phi_1(x_1)\dots\phi_n(x_n)\rangle = \frac{\langle 0|\,\mathrm{T}\left\{\hat{S}\cdot\hat{\phi}_{I1}(x_1)\dots\hat{\phi}_{In}(x_n)\right\}|0\rangle}{\langle 0|\,\mathrm{T}\,\hat{S}\,|0\rangle},$$

where the *scattering operator* $\hat{S}$ is defined as

$$\hat{S} = \lim_{T\to\infty}\hat{U}(T,-T) = \mathrm{T}\,\exp\left\{-i\int\limits_{-\infty}^{+\infty}dt\,\hat{H}_I(t)\right\}.$$

From this, it is clear that the normalization condition can be reformulated as a trivial equation on the empty correlation bracket: $\langle 1\rangle = 1$.
Both the numerator and the denominator of the fraction above contain time-ordered exponentials, which can be Tailor-expanded to give the exact same objects (correlations), but in the context of the corresponding *free quantum field theory.* Therefore we have reduced the problem of calculation of correlations in the interacting theory to the same problem in the corresponding free theory, which *is already solved by means of Wick's theorem.* But since we are working with formal operator Taylor series, the only way we can actually perform computations is to truncate this series at some finite index. But this in fact is what one would generally expect from perturbative computations after all.

Feynman rules for ϕ^4 theory

The ϕ^4 theory is a toy model of an interacting quantum field theory. It is defined by the action

$$S[\phi] = \int d^4x\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right),$$

where $\lambda \ll 1$ is a small dimensionless coupling constant which measures the strength of the interaction.

Its Hamiltonian can be split in the free part (which describes a free scalar field) and a small perturbation:

$$\hat{H}_0 = \int d^3x\left(\frac{1}{2}:\hat{\pi}^2: + \frac{1}{2}:\left(\nabla\hat{\phi}\right)^2: + \frac{1}{2}m^2:\hat{\phi}^2:\right);$$

$$\hat{H}_I = \int d^3x\left(\frac{\lambda}{4!}:\hat{\phi}^4:\right).$$

Note that there is an ordering ambiguity here. We have chosen both parts to be normal-ordered in order to throw away the (unphysical) diverging vacuum energy ($:\dots:$ stands for normal-ordering). Also, $\hat{\phi}$ and $\hat{\pi}$ here are quantum operators in the interaction picture, hence they obey the Heisenberg equation with $\hat{H}_0$ instead of $\hat{H}$ (we drop the I subscript in this section since we no longer need Heisenberg picture).

Lets calculate the scattering operator:

$$\hat{S} = \mathrm{T}\,\exp\left\{-i\int\limits_{-\infty}^{+\infty}dt\int d^3x\left(\frac{\lambda}{4!}:\hat{\phi}(\vec{x},t)^4:\right)\right\} = \mathrm{T}\,\exp\left\{\frac{-i\lambda}{4!}\cdot\int d^4x\,:\hat{\phi}(x)^4:\right\} =$$

$$= \hat{1} + \frac{-i\lambda}{4!}\int d^4x\,:\hat{\phi}(x)^4: + \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2\int d^4x\int d^4y\,\mathrm{T}\left\{:\hat{\phi}(x)^4::\hat{\phi}(y)^4:\right\} +$$

$$+\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4x\int d^4y\int d^4z\,\mathrm{T}\left\{:\hat{\phi}(x)^4::\hat{\phi}(y)^4::\hat{\phi}(z)^4:\right\} + \dots.$$

This is a perturbative expansion for $\hat{S}$ in powers of the coupling constant λ.

We are ultimately interested in calculating correlations. Lets compute an n-th order correlation:

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = N^{-1}\cdot\langle 0|\,\mathrm{T}\left\{\hat{S}\cdot\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\right\}|0\rangle,$$

where $N = \langle 0| \hat{S} |0\rangle$ is the normalization factor. We now use the perturbative expansion for the scattering operator and obtain a perturbative series for the correlation:

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle = N^{-1} \cdot \Bigg(\langle 0| \, \mathrm{T} \left\{ \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} |0\rangle + \frac{-i\lambda}{4!} \int d^4x \; \langle 0| \, \mathrm{T} \left\{ : \hat{\phi}(x)^4 : \, \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} |0\rangle +$$

$$+ \frac{1}{2!} \left(\frac{-i\lambda}{4!} \right)^2 \int d^4x \int d^4y \; \langle 0| \, \mathrm{T} \left\{ : \hat{\phi}(x)^4 :: \hat{\phi}(y)^4 : \, \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} |0\rangle + \ldots \Bigg).$$

Since $\hat{\phi}$ are interaction-picture operators and therefore obey the free theory equations of motion, each term in this series can be calculated with help of Wick's theorem. For example,

$$\mathrm{T} \left\{ \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} = : \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) : + (\text{all possible contractions}).$$

Therefore, since all normal-ordered operators have zero vacuum-expectation value,

$$\langle 0| \, \mathrm{T} \left\{ \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} |0\rangle = (\text{all possible full contractions}).$$

More generally,

$$\langle 0| \, \mathrm{T} \left\{ : \hat{\phi}(x)^4 : \, \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} |0\rangle =$$

$$= (\text{all possible full contractions; fields inside :...: can not be contracted with each other}).$$

To each such contraction we associate a graph, called a *Feynman diagram*. Vertices of the graph are all 4-valent and correspond to the $: \hat{\phi}(x)^4 :$ insertions (which should be contracted with 4 different vertices thus giving 4-valency). Edges of the graph correspond to contractions, which are equal to the propagators of the free theory.

Now we can calculate all terms in our perturbative expansions without doing tedious Wick-theorem calculations explicitly. In order to do so, we introduce Feynman rules:

Position-space Feynman rules for ϕ^4 theory:

- The n-th order correlation is proportional (not equal, because of the normalization factor N) to the sum of all possible (not necessarily connected) Feynman diagrams with exactly n external lines.
- To each Feynman diagram we assign an analytical expression.
- To each vertex we assign a numerical factor and an integral over the spacetime position of the vertex:

  $$-i\lambda \int d^4x.$$

  Therefore, vertexes represent interactions of elementary particles. We can restrict ourselves to diagrams with less or equal than n vertices (for some natural n) since this is precisely the $O(\lambda^n)$ approximation of the perturbative series.
- To each edge we assign a propagator of the free theory. Therefore, edges represent propagating elementary particles.
- Almost all combinatorical factors cancel each other out. For example, there is exactly 4! ways to rearrange the four $\hat{\phi}(x)$ operators in the insertion (and thus 4! ways to contract them with other operators in such a way that it will render the same diagram). This factor precisely cancels the $1/4!$ factor which came here all the way from the classical action in each vertex. There is also $n!$ ways to relabel n vertexes in the diagram, which cancels the $1/n!$ factor from the Tailor series of the exponential. The only remaining factors are symmetry factors which arise in diagrams which edges can be relabeled so that the resulting diagram is isomorphic to the original one. The inverse symmetry factor is equal to the number of such relabelings.