FAQ on Conformal Field Theory

Classical theory and conformal algebra

What are conformal transformations?

Consider a generally-covariant field theory on the (pseudo) Riemannian spacetime manifold with background metric $g_{\mu\nu}(x)$. By definition, generally-covarient theories possess a large group of geometrical symmetries — arbitrary coordinate transformations or *diffeomorphisms*.

Now consider a *Weyl transformation* parametrized by a scalar field $\omega(x)$: by definition, it changes the background metric to be

$$g_{\mu\nu}(x) \to e^{\omega(x)} \cdot g_{\mu\nu}(x)$$

and leaves the dynamical fields untouched. Classical field theory is Weyl-invariant if and only if its stress-energy tensor is traceless:

$$\delta S = \int d^d x \, \frac{\partial S}{\partial g^{\mu\nu}(x)} \cdot \delta g^{\mu\nu}(x) = \int d^d x \sqrt{g} \, T_{\mu\nu}(x) \delta g^{\mu\nu}(x) =$$

$$= \int d^d x \sqrt{g} \, T_{\mu\nu}(x) \left(-\delta\omega(x) \cdot g^{\mu\nu}(x) \right) = - \int d^d x \sqrt{g} \delta\omega(x) \cdot T_\mu{}^\mu(x),$$

$$\delta S = 0 \quad \Longleftrightarrow \quad T_\mu{}^\mu(x) = 0.$$

By definition, a *conformal transformation* is a diffeomorphism which changes the metric field $g_{\mu\nu}(x)$ in such a way that this change can be compensated by a Weyl transformation, bringing $g_{\mu\nu}(x)$ to its original value.

How can I get a feel for conformal generators?

The geometrical objects generating diffeomorphisms are vector fields (sections of the tangent bundle) on the spacetime manifold. They act on the geometrical objects (fields) through the Lie derivative. For example, the action of the diffeomorphism generator $f^\mu(x)$ on the tensor field $\Phi^{b_1 b_2 \ldots}_{a_1 a_2 \ldots}$ is

$$\mathcal{L}_f \Phi^{b_1 b_2 \ldots}_{a_1 a_2 \ldots} = f^\mu \partial_\mu \Phi^{b_1 b_2 \ldots}_{a_1 a_2 \ldots} + \sum_k \Phi^{b_1 b_2 \ldots}_{a_1 \ldots \mu \ldots} \cdot \partial_{a_k} f^\mu - \sum_k \Phi^{b_1 \ldots \mu \ldots}_{a_1 a_2 \ldots} \cdot \partial_\mu f^{b_k}.$$

The geometrical objects generating Weyl transformations are scalar fields. They act trivially on dynamical fields and only perturb the metric:

$$\mathcal{L}_\omega \Phi^{b_1 b_2 \ldots}_{a_1 a_2 \ldots} = 0,$$

$$\mathcal{L}_\omega g_{\mu\nu} = \omega(x) \cdot g_{\mu\nu}.$$

Since conformal transformations are diffeomorphisms compensated by the Weyl rescaling, they are generated by vector fields of the special kind — *conformal vector fields* satisfying

$$0 = \mathcal{L}_{\text{conformal}} \circ g_{\mu\nu}(x) = \mathcal{L}_f g_{\mu\nu}(x) + \mathcal{L}_\omega g_{\mu\nu}(x) =$$

$$= f^\sigma(x) \partial_\sigma g_{\mu\nu}(x) + \partial_\mu f^\sigma(x) \cdot g_{\sigma\nu}(x) + \partial_\nu f^\sigma(x) \cdot g_{\mu\sigma}(x) + \omega(x) \cdot g_{\mu\nu}(x) =$$

$$= \nabla_\mu f_\nu(x) + \nabla_\nu f_\mu(x) + \omega(x) \cdot g_{\mu\nu}(x) = 0$$

for some $\omega(x)$. By taking the trace, we can see that

$$\omega(x) = -\frac{2}{d} \nabla_\sigma f^\sigma(x).$$

So conformal transformations are generated by conformal vector fields which satisfy the *conformal Killing equation*:

$$\nabla_\mu f_\nu(x) + \nabla_\nu f_\mu(x) - \frac{2}{d} \nabla_\sigma f^\sigma(x) \cdot g_{\mu\nu}(x) = 0.$$

Why is the conformal symmetry special in 2 dimensions?

In flat spacetime, the conformal Killing equation takes the following form:

$$\partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{d}\,(\partial f)\,\delta_{\mu\nu} = 0.$$

For $d > 2$ its solutions form a finite-dimensional vector space — the conformal algebra, which can be shown to be isomorphic to $SO(d+1,1)$. However, for $d = 2$, the conformal Killing equation becomes

$$\partial_1 f_1 = \partial_2 f_2, \quad \partial_1 f_2 = -\partial_2 f_1,$$

which is exactly the Cauchy-Riemann condition for the complexified function

$$f(z) = f_1(x_1 + ix_2) + if_2(x_1 + ix_2)$$

to be analytic. Thus, in two dimensions conformal transformations are generated by an infinite-dimensional algebra of analytic functions on the complex plane.

Whats a local conformal algebra?

The mentioned generators can be expanded in the vicinity of some point P which we will assign a complex coordinate $z = 0$. Conformal transformations are generated by analytical vector fields $f(z)$ which can be expanded into the Laurent series

$$f(z) = \sum_n c_n z^{n+1}\partial_z = \sum_n c_n l_n,$$

where we have introduced a basis for the generators of conformal algebra

$$l_n = z^{n+1}\partial_z.$$

We now proceed with algebraic relations. Those are given by the adjoint action of vector fields on themselves, which is a special case of the action on tensors and is given by the special case of Lie derivative — the Lie bracket:

$$\begin{aligned} [f,h] &= \mathcal{L}_f h = (f^\mu \partial_\mu h^\nu - h^\mu \partial_\mu f^\nu)\,\partial_\nu, \\ [l_m, l_n] &= \left(z^{m+1}\partial_z z^{n+1} - z^{n+1}\partial_z z^{m+1}\right)\partial_z = \\ &= \left((n+1)z^{n+m+1} - (m+1)z^{n+m+1}\right)\partial_z = \\ &= [l_m, l_n] = (n-m)l_{m+n}. \end{aligned}$$

This algebra is known as the local conformal algebra, or the Witt algebra.

How is the reality condition imposed?

Actually, we can promote the coordinates x^1 and x^2 to complex numbers. Then, another choice of coordinates

$$\begin{cases} z = x^1 + ix^2 \\ \bar{z} = x^1 - ix^2 \end{cases}$$

will give us two compies of the local conformal algebra:

$$\begin{cases} l_n = z^{n+1}\partial_z \\ \bar{l}_n = \bar{z}^{n+1}\partial_{\bar{z}} \end{cases}$$

At the end of the day, however, we want to impose the reality conditions:

$$x^1 \in \mathbb{R}, \quad x^2 \in \mathbb{R},$$

or, equivalently,

$$\bar{z} = z^*.$$

Note that z and $\bar{z}$ are two independent complex coordinates while z^* denotes complex conjugation. The vector fields generating local conformal transformations on the real-valued plane are thus

$$\begin{cases} l_{(1),n} = l_n + \bar{l}_n \\ l_{(2),n} = i\left(l_n - \bar{l}_n\right) \end{cases}$$

How does the global topology affect the conformal algebra?

The local conformal algebra always acts in the vicinity of some point P. If we want to construct the algebra of global conformal transformations, we have to take into account the topological obstructions.

Consider, for example, the Euclidean cylinder, which can be mapped conformally onto the complex plane which a single puncture, say, at $P : z_P = 0$ (the two sides of the cylinder are mapped onto P and ∞). This case is important for string theory since the worldsheet of the colsed string has the same topology. In this case the global conformal algebra is equivalent to the local conformal algebra (i.e. two copies of the Witt algebra) since all analytical fields with poles at $z = 0$ and $z = \infty$ allow the Laurent expansion which converges everywhere except these two points.

Now consider the Euclidean plane, which is the most straightforward background for the field theory to be defined on. In this case we require the global conformal generators not to have singularities at $z = 0$. Thus, we can only take l_n and $\bar{l}_n$ for $n \geq -1$. It can be checked that these generators close a subalgebra of global conformal transformations on the plane.

Finally, consider the Riemann sphere (a compactified complex plane with a special point $z = \infty$). In this case we require the genreators to be regular not only at $z = 0$, but also at $z = \infty$. By passing to the inverse coordinate $w = 1/z$,

$$l_n = z^{n+1}\partial_z = (-w)^{1-n}\partial_w,$$

and requiring that l_n does not have singularities at $w = 0$, we find that $n \leq 1$. Thus, the six generators of the global conformal algebra on the Riemann sphere are

$$\left\{l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1\right\}.$$

Note that these generate the group $SL(2,\mathbb{C}) \simeq SO(3,1)$, which we would have got if we haven't noticed that conformal algebra is special in 2 dimensions.

How does Weyl/conformal invariance manifest itself classically?

Since conformal transformations preserve the metric (by definition, the diffeomorphism is followed by a compensating Weyl rescaling), classical conformal-invariant theories come from putting Weyl-invariant and generally-covariant theories on the flat background.

For example, consider the classical Klein-Gordon field theory:

$$S[\phi] = -\frac{1}{2}\int d^dx\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

Under the Weyl rescaling,

$$\begin{aligned}
g_{\mu\nu} &\to e^{\omega(x)}g_{\mu\nu}, \\
g^{\mu\nu} &\to e^{-\omega(x)}g^{\mu\nu}, \\
\det g_{\mu\nu} &\to e^{d\cdot\omega(x)}\det g_{\mu\nu}, \\
\sqrt{g} &\to e^{\frac{d}{2}\cdot\omega(x)}\sqrt{g}, \\
\sqrt{g}g^{\mu\nu} &\to e^{\left(\frac{d}{2}-1\right)\omega(x)}\sqrt{g}g^{\mu\nu}.
\end{aligned}$$

We see that unless the ϕ field transforms non-trivially, the theory is Weyl-invariant only in 2 spacetime dimensions.

Another example is the Maxwell theory:

$$S[A] = -\frac{1}{4}\int d^dx\sqrt{g}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta},$$

$$\sqrt{g}g^{\mu\alpha}g^{\nu\beta} \to e^{\left(\frac{d}{2}-2\right)\omega(x)}\sqrt{g}g^{\mu\alpha}g^{\nu\beta}.$$

We see that Maxwell theory is actually Weyl-invariant in 4 spacetime dimensions, which is why the electromagnetic stress-energy tensor is always traceless.

Weyl invariance does not allow mass or cosmological terms in the action, since $\sqrt{g}$ will not be compensated by a required number of inverse metric tensors.

Quantum CFT

Is quantum theory Weyl-invariant?

In order for Weyl transformations to describe the redundancy in the quantum theory, both the action and the functional measure have to be Weyl-invariant. In most of the cases, the functional measure is not Weyl-invariant, spawning the *Weyl anomaly*.

This can be seen by writing down the scalar product on the tangent space of field configurations, from which the regularized functional measure is constructed. For example, for the Klein-Gordon theory in 2 spacetime dimensions:

$$\|\delta\phi\|^2 = \int d^2x\sqrt{g}\,\delta\phi^2,$$

which is obviously not Weyl-invariant. The same happens in Maxwell and Yang-Mills theories in 4 dimensions.

How do we compute the anomaly?

Probably the most direct and beautiful method is to regularize the path integral measure and derive its variation under the Weyl transformation. This can be done with help of heat-kernel regularization. For more details, see the first section of this paper.

In the conformal gauge

$$g_{\mu\nu}(x) = e^{\omega(x)}\delta_{\mu\nu},$$

the resulting expression reads

$$\int D\phi_\omega = \int D\phi_0 e^{-cS_{\text{Li}}[\omega]},$$

where $D\phi_0$ is the flat functional measure (which is by definition Weyl-invariant). Thus, the dependence of the functional measure $D\phi$ on the Weyl factor $\omega(x)$ is described by the Liouville action

$$S_{\text{Li}}[\omega] = \frac{1}{48\pi}\int d^2x\left[\mu(\epsilon)^2 e^{\omega(x)} + \frac{1}{2}(\partial\omega)^2\right].$$

The constant c in front of the Liouville action is called the *central charge* of the theory. Its numeric value is determined by the properties of the functional measure. For example, for the scalar field with D components, $c = D$.

The Liouville action contains a divergent cosmological constant term $\mu^2(\epsilon)$. The renormalization condition is simply that the partition function

$$Z = \int D\phi_\omega e^{-S[\phi]} = \int D\phi_0 e^{-(S[\phi]+cS_{\text{Li}}[\omega])}$$

is independent of the regularizer in the limit $\epsilon \to 0$ (and thus can be taken to be $Z = 1$).

The conformal anomaly has shifted from the non-invariance of the functional measure to the non-invariance of the Liouville action. Thus, instead of dealing with the functional measure directly, we have introduced an auxiliary conformal-invariant measure $D\phi_0$ at the cost of adding the Liouville counter-terms which break the conformal-invariance of the action.

What is the trace anomaly?

Consider the stress-energy tensor for the full action for $D = c$ independent scalar fields in the conformal gauge:

$$S[\phi^a] = \int d^2x\, e^{\omega(x)}\left[\frac{1}{2}e^{-\omega(x)}(\partial\phi)^2 + \frac{c\mu(\epsilon)^2}{48\pi} + \frac{c}{96\pi}\cdot e^{-\omega(x)}(\partial\omega)^2\right],$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2\,\delta_{\mu\nu} - \frac{c\mu(\epsilon)^2}{48\pi}e^{\omega(x)}\delta_{\mu\nu} - \frac{c}{96\pi}(\partial\omega)^2\delta_{\mu\nu}.$$

In flat space $\omega(x) = 0$ and $\mu(\epsilon)$ is fixed by the renormalization condition $Z(\epsilon \to 0) = \text{const}$, which is equivalent to taking $\langle T_{\mu\nu}\rangle = 0$. Thus, on the general background,

$$\langle T_{\mu\nu}\rangle = -\frac{c}{96\pi}\cdot(\partial\omega)^2\delta_{\mu\nu}.$$

The trace of this full stress-energy tensor is equal to

$$\langle T_\mu{}^\mu \rangle = e^{-\omega(x)}\delta^{\mu\nu}\langle T_{\mu\nu}\rangle = -\frac{c}{48\pi}\cdot e^{-\omega}(\partial\omega)^2 = \frac{c}{24\pi}R,$$

where R is the scalar curvature of the metric. This equation is known as the trace-anomaly relation. Since the trace does not vanish in curved space anymore, we conclude (again) that the conformal symmetry is broken in the quantum theory.

But flat-space CFT is not anomalous, right?

Not exactly. The action still contains the cosmological term which breaks conformal invariance:

$$S[\phi] = \int d^2x\left[\frac{1}{2}(\partial\phi)^2 + \frac{c\mu(\epsilon)^2}{48\pi}\right].$$

But it is possible to resurrect conformal symmetry on the level of correlation functions at the cost of modifying the transformation properties of fields. Or, in the canonical language, it is possible to assign unitary quantum operators corresponding to conformal transformations at the cost of losing classical geometric intuition for the transformation properties of fields.

How to derive the transformation properties of quantum operators?

Consider a theory of $D = c$ scalar fields on the plane in complex coordinates:

$$S[\phi] = -\frac{1}{8\pi}\int d^2z\left[\partial_z\phi\cdot\partial_{\bar z}\phi + c\mu^2\right],$$

where we have rescaled the bare coupling μ (after all, it is fixed by the renormalization condition $\langle T_{\mu\nu}\rangle = 0$).

Consider a conformal transformation generated by l_n:

$$z \to z' = z + \epsilon\cdot z^{n+1}.$$

The spacetime measure picks up the Jackobian factor:

$$d^2z' = d^2z\left(1 + (n+1)\epsilon z^n\right).$$

So the infinitesimal change of the action is given by the diverging integral

$$\delta S = -\frac{(n+1)c\mu^2}{8\pi}\int d^2z\cdot z^n\epsilon.$$

Now lets allow the infinitesimal parameter ϵ to depend on z and $\bar z$:

$$\delta S = -\frac{(n+1)c\mu^2}{8\pi}\int d^2z\cdot z^n\epsilon(z,\bar z) + \frac{1}{2\pi}\int d^2z\cdot J^\mu\partial_\mu\epsilon(z,\bar z) =$$

$$= -\frac{(n+1)c\mu^2}{8\pi}\int d^2z\cdot z^n\epsilon(z,\bar z) - \frac{1}{2\pi}\int d^2z\epsilon(z,\bar z)\cdot\partial_\mu J^\mu,$$

where

$$T_{zz} = \partial_z\phi\cdot\partial_z\phi,$$
$$T_{\bar z\bar z} = \partial_{\bar z}\phi\cdot\partial_{\bar z}\phi,$$
$$T_{z\bar z} = \frac{c\mu^2}{8\pi} = \mathrm{const}(z,\bar z),$$
$$J_z = z^{n+1}T_{zz} + \mathrm{const}, \quad J_{\bar z} = \mathrm{const}.$$

is the Noether current associated with the conformal transformation.

Lets take $\epsilon(z,\bar z)$ to be 1 in the close vicinity of some point z_0 and 0 everywhere else. Then the first term in the expression for δS will fade away, because it is proportional to the (extremally small) area of the mentioned vicinity. Then

$$\delta S = -\frac{1}{2\pi}\int_\epsilon d^2z\left(\partial_z J^z + \partial_{\bar z}J^{\bar z}\right) = -\frac{1}{2\pi i}\int_{\partial\epsilon}\left(J_z dz - J_{\bar z}d\bar z\right).$$

Consider now the quantum expectation of some functional $\Omega[\phi]$:

$$\langle\Omega\rangle = \int D\phi e^{-S[\phi]}\Omega[\phi].$$

Since we use the conformal-invariant measure (we have shifted the anomaly to the action), the transformation of a dummy integration variable ϕ does not change the expectation. Thus,

$$\langle\delta\Omega\rangle = \langle\Omega\cdot\delta S\rangle\,.$$

Consider

$$\Omega[\phi] = \Omega_0(z_0)\cdot\Omega_1(z_1)\cdot\ldots$$

with $z_1, z_2, \ldots$ outside of ϵ. Then

$$\delta\Omega = \delta\Omega_0(z_0)\cdot\Omega_1(z_1)\cdot\Omega_2(z_2)\cdot\ldots$$

and the conformal Ward identity holds:

Why do all operators commute in CFT?

Just like in any quantum theory, operators don't commute. The reason that we write $\hat{a}\hat{b} = \hat{b}\hat{a}$ in CFT lies in the fact that operator strings are considered automatically time-ordered. This notation is specific to CFT calculations.

So, for example, the commutator of two operators at time t is equal to

$$\left[\hat{a},\hat{b}\right](t) = \lim_{\epsilon\to+0}\left(\hat{a}(t+\epsilon)\hat{b}(t) - \hat{a}(t)\hat{b}(t+\epsilon)\right).$$

What is OPE?

// TODO

How does the stress-energy operator transform?

// TODO

How to derive the Virasoro algebra?

// TODO

Why has the structure of the Lie algebra changed?

// TODO

Why does string theory require 26 dimensions?

// TODO