Classical linearized gravity

We approximate the general-relativistic metric $g_{\mu\nu}(x)$ around the flat Minkowski background:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon h_{\mu\nu}(x) + O(\epsilon^2),$$

where $h_{\mu\nu}$ lies within the traceless symmetric representation of the Poincare group. The inverse metric is given by

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \epsilon h^{\mu\nu}(x) + O(\epsilon^2),$$

which can be checked to give

$$g_{\mu\nu}g^{\nu\sigma} = \eta_{\mu\nu}\eta^{\nu\sigma} + \epsilon\left(h_{\mu\nu}\eta^{\nu\sigma} - \eta_{\mu\nu}h^{\nu\sigma}\right) + O(\epsilon^2) = \delta^\sigma_\mu + O(\epsilon^2).$$

Curvature tensors

According to General Relativity, the gravitational field is determined by the components of the affine connection $\Gamma^\sigma_{\mu\nu}$, which can be expanded to the first order in ϵ:

$$\Gamma^\sigma_{\mu\nu} = \frac{g^{\sigma\tau}}{2}\left(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}\right) =$$

$$= \frac{\eta^{\sigma\tau} + O(\epsilon)}{2}\left(\epsilon h_{\nu\tau,\mu} + \epsilon h_{\mu\tau,\nu} - \epsilon h_{\mu\nu,\tau}\right) =$$

$$= \Gamma^\sigma_{\mu\nu} = \frac{\epsilon}{2}\left(h^\sigma_{\nu,\mu} + h^\sigma_{\mu,\nu} - h^{;\sigma}_{\mu\nu}\right) + O(\epsilon^2).$$

Now we can calculate the Ricci tensor:

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} =$$

$$= \frac{\epsilon}{2}\left(h^\rho_{\nu,\mu\rho} + h^\rho_{\mu,\nu\rho} - h^{;\rho}_{\mu\nu,\rho} - h^\rho_{\rho,\mu\nu} - h^\rho_{\mu,\rho\nu} + h^{;\rho}_{\mu\rho,\nu}\right) + O(\epsilon^2) =$$

$$= R_{\mu\nu} = \frac{\epsilon}{2}\left(h^\rho_{\nu,\mu\rho} + h^{;\rho}_{\mu\rho,\nu} - h^{;\rho}_{\mu\nu,\rho} - h^\rho_{\rho,\mu\nu}\right) + O(\epsilon^2).$$

$$\Gamma^\nu_{\mu\nu} = \frac{\epsilon}{2}\left(h^\nu_{\nu,\mu} + h^\nu_{\mu,\nu} - h^{;\nu}_{\mu\nu}\right)$$

Gauge invariance

The vacuum Einstein's equations are (to the order of ϵ):

$$R_{\mu\nu} = \frac{\epsilon}{2}\left(h^\rho_{\nu,\mu\rho} + h^{;\rho}_{\mu\rho,\nu} - h^{;\rho}_{\mu\nu,\rho} - h^\rho_{\rho,\mu\nu}\right) = 0.$$

They are linear in $h_{\alpha\beta}(x)$ and are generated by the differential operator

$$-\frac{2}{\epsilon}R_{\mu\nu} = \hat{\Theta}^{\alpha\beta}_{\mu\nu} \circ h_{\alpha\beta} = 0,$$

$$\hat{\Theta}^{\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu\Box + \eta^{\alpha\beta}\partial_\mu\partial_\nu - \left(\delta^\alpha_\nu\partial_\mu + \delta^\alpha_\mu\partial_\nu\right)\eta^{\beta\rho}\partial_\rho.$$

The differential operator $\hat{\Theta}^{\alpha\beta}_{\mu\nu}$ is singular and does not have an inverse. This is because of the gauge invariance of the theory which comes from the diffeomorphism invariance of General Relativity.

Consider an infinitesimal generator of a diffeomorphism — a vector field $f^\mu(x)$. The infinitesimal variation of the metric is given by the Lie derivative

$$\delta g_{\mu\nu} \sim \left(\mathcal{L}_f g\right)_{\mu\nu} = f^\sigma\partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu f^\sigma + g_{\mu\sigma}\partial_\nu f^\sigma.$$

To the zeroth order in ϵ,

$$\delta h_{\mu\nu}(x) \sim \delta g_{\mu\nu}(x) \sim \eta_{\sigma\nu}\partial_\mu f^\sigma + \eta_{\mu\sigma}\partial_\nu f^\sigma + O(\epsilon),$$

$$\delta h_{\mu\nu} \sim \partial_\mu f_\nu + \partial_\nu f_\mu.$$

Thus, vector fields generate the spacetime diffeomorphisms, which, after the weak-field expansion is performed, become the gauge transformations acting on the linearized gravitational field $h_{\mu\nu}(x)$.

Harmonic gauge

We can fix the gauge transformations by imposing the harmonic condition:

$$h_{\mu\nu}^{;\nu} = \frac{1}{2}\eta^{\sigma\tau}h_{\sigma\tau,\mu}.$$

It is possible to show that when this condition holds, the last two thirds of the expression for $\hat{\Theta}_{\mu\nu}^{\alpha\beta}$ vanish:

$$\eta^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - \left(\delta_\nu^\alpha\partial_\mu + \delta_\mu^\alpha\partial_\nu\right)\eta^{\beta\rho}\partial_\rho h_{\alpha\beta} =$$

$$= \eta^{\sigma\tau}h_{\sigma\tau,\mu\nu} - h_{\nu,\mu\rho}^{\rho} - h_{\mu,\nu\rho}^{\rho} = \partial_\mu\left(\frac{1}{2}\eta^{\sigma\tau}h_{\sigma\tau,\nu} - h_{\nu,\rho}^{\rho}\right) +$$

$$+\partial_\nu\left(\frac{1}{2}\eta^{\sigma\tau}h_{\sigma\tau,\mu} - h_{\mu,\rho}^{\rho}\right) = \partial_\mu 0 + \partial_\nu 0 = 0.$$

Thus, in the harmonic gauge,

$$\left(\hat{\Theta}_{\text{harmonic}}\right)_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha\delta_\nu^\beta\Box.$$

The harmonic gauge is analogous to the Lorentz gauge from electromagnetism, where the Maxwell differential operator takes the form

$$\left(\hat{\Theta}_{\text{Lorentz}}\right)_\mu^\alpha = \delta_\mu^\alpha\Box.$$

Gravitational waves

By analogy with the Maxwell theory, in the harmonic gauge the solutions of the vacuum Einstein equations are superpositions of plane gravitational waves of form

$$h_{\mu\nu}(x) = \zeta_{\mu\nu}(p)\cdot e^{ip_\mu x^\mu}.$$

Note that we couldn't've used the Fourier transform directly in General Relativity since GR is background-independent and there is no preferred background spacetime with a global coordinate patch $\{x^\mu\}$.

We must have

$$p^2 = 0$$

in order to satisfy the wave equation in the harmonic gauge. Thus, gravitons are massless. The polarization tensor $\zeta_{\mu\nu}(p)$ must satisfy the Fourier-transformed condition for the harmonic gauge:

$$p^\nu\zeta_{\mu\nu} - \frac{1}{2}p_\mu\eta^{\sigma\tau}\zeta_{\sigma\tau} = 0,$$

which singles out the two polarizations of the gravitational wave.

The graviton propagator in the harmonic gauge is given by the inverse of the operator $\hat{\Theta}\cdot\delta^{(4)}(x-y)$:

$$\Delta_{\mu\nu|\alpha\beta}(p) = \frac{-i\eta_{\mu\alpha}\eta_{\nu\beta}}{p^2 + i0}.$$