Classical Electrodynamics

The classical laws gouverning the electromagnetic forces have been formulated by James Clerk Maxwell in the form of four differential equations in partial derivatives:

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi\rho, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times \vec{B} = \frac{1}{c}\left(4\pi\vec{j} + \frac{\partial}{\partial t}\vec{E}\right). \end{cases}$$

Here we assume $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ to be the electric and magnetic fields (the basic variables of the electrodynamics theory) distributed over the 3-dimensional flat Euclidean space with coordinates $\vec{x} = (x, y, z)$ and changing with time t.

The sources of the electromagnetic field are the charge density $\rho(\vec{x},t)$ and the current density $\vec{j}(\vec{x},t)$. These modify the vacuum electromagnetic field by generating a perturbation, which we observe as electric and magnetic forces.

The constant c is the speed of light, which has to do with electromagnetism since light itself is a wave of electromagnetic radiation. Just like that, Maxwell's theory not only unifies electricity with magnetism, but also explains the phenomenon of light.

The covariant form of Maxwell's equations

Maxwell equations are covariant under Lorentz transformations (this mathematical fact has originally lead Einstein to Special Relativity). It could be seen as we rewrite the equations in the manifestly covariant form. We introduce the 4-dimensional spacetime coordinate system

$$x = \left\{x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z\right\} = \{ct,\ \vec{x}\}.$$

Both $\vec{E}$ and $\vec{B}$ can be unified into a rank-2 antisymmetric electromagnetic field strength tensor:

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

After raising both indices with the metric tensor $g_{\mu\nu}(x) = \mathrm{diag}\,(1, -1, -1, -1)$ we get

$$F^{\mu\nu}(x) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Now we claim that Maxwell equations can be written in the manifestly Lorentz-covariant form:

$$\begin{cases} \varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0, \\ \partial_\mu F^{\mu\sigma} = 4\pi j^\sigma \end{cases}$$

with the source 4-vector current

$$j^\mu = \left\{\rho,\ \frac{\vec{j}}{c}\right\}$$

and the totally antisymmetric Levi-Civita symbol defined by

$$\varepsilon_{0123} = -\varepsilon^{0123} = 1.$$

- The first equation for $\sigma = 0$ is equivalent to $\vec{\nabla} \cdot \vec{B} = 0$ (the second Maxwell equation).
- The first equation for $\sigma \in \{1, 2, 3\}$ is equivalent to $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}$ (the third Maxwell equation).
- The second equation for $\sigma = 0$ is equivalent to $\vec{\nabla} \cdot \vec{E} = 4\pi j^0 = 4\pi\rho$ (the first Maxwell equation).
- The second equation for $\sigma \in \{1, 2, 3\}$ is equivalent to $\vec{\nabla} \times \vec{B} = 4\pi\frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$ (the fourth Maxwell equation).

Electromagnetic gauge potential

We can further simplify the equations for electromagnetism by noting that the first covariant equation

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0$$

is automatically satisfied by an exact rank-2 differential form

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

The field equations are then just the second covariant equation rewritten for the gauge potential $A_\mu(x)$:

$$\partial_\mu \left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right) = 4\pi j^\sigma$$

or

$$\left(\Box\delta^\sigma_\mu - \partial_\mu\partial^\sigma\right) A^\mu = 4\pi j^\sigma.$$

This equation contains all the information given originally in the Maxwell equations, however, it can not be used to determine uniquely the gauge potential $A_\mu(x)$ given the source 4-vector $j^\mu(x)$.

- The technical difficulty which prohibits as from solving for A_μ can be seen if we try to write down the Green's function for the linear differential operator

  $$\hat{\Theta}^\sigma_\mu = \Box\delta^\sigma_\mu - \partial_\mu\partial^\sigma.$$

  If we try to do the Fourier transform (by trading $\partial_\mu \leftrightarrow ik_\mu$) and calculate the inverse of the resulting matrix

  $$\tilde{\Theta}^\sigma_\mu(k) = k_\mu k^\sigma - k^2\delta^\sigma_\mu,$$

  we immediately discover that $\det\tilde{\Theta}(k) = 0$ and no inverse matrix exists. Thus, $\hat{\Theta}^\sigma_\mu$ is singular (does not have a well-defined Green's function).

- The conceptual difficulty which prohibits as from solving for A_μ is the fact that multiple choices of the gauge potential render the same electromagnetic field $F_{\mu\nu}$. In fact, consider a change of the gauge potential of the form

  $$A'_\mu = A_\mu + \partial_\mu\omega(x)$$

  with any scalar function of spacetime $\omega(x)$ — a gradient transformation. The electromagnetic field strength is equal to

  $$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu + \partial_\mu\partial_\nu\omega(x) - \partial_\nu A_\mu - \partial_\mu\partial_\nu\omega(x) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

  Thus, a gradient transformation does not change the electromagnetic field strength. It turns out that it is exactly these transformations which render the field equation generated by the differential operator $\hat{\Theta}$ unsoluble for A_μ. In fact, this equation allows for an infinite set of solutions related by gradient transformations, and all of them correspond to the same electromagnetic field strength $F_{\mu\nu}$. In other words, the equation is capable of predicting $F_{\mu\nu}$, but not A_μ, since the gauge potential is unphysical.

Gauge fixing in classical theory

In case we still wish to proceed with solving for A_μ, we have to add an extra equation which would pick a specific field configuration out of all the equivalent ones (related by gradient transformations).

The Lorentz gauge condition is given by

$$\partial_\mu A^\mu = 0$$

In the Lorentz gauge the field equation for the gauge potential simplifies dramatically:

$$\Box A^\mu = 4\pi j^\mu.$$

It can be solved then by means of the Green's function of the Klein-Gordon operator $\Box$:

$$G(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik_\mu(x^\mu - y^\mu)}}{(k+i0)^2},$$

$$A^\mu(x) = 4\pi\int d^4y\, G(x,y) j^\mu(y) + A^\mu_{\text{free}}(x)$$

where $A^\mu_{\text{free}}(x)$ is a solution of the free field equation

$$\Box A^\mu_{\text{free}}(x) = 0.$$