Barrett-Crane spinfoams

This post follows my previous post on how to quantize the topological BF-theory in the background-independent fashion using the spinfoam formalism. Here we will try adopting the spinfoam techniques to quantize 4-dimensional General Relativity.

The models described here are quantizations of the Riemannian 4-dimensional General Relativity. In the frame-connection (Palatini) formalism it has the gauge group $SO(4)$. The indices of the defining 4-dimensional representation of $SO(4)$ are capital latin and are raised and lowered with the Kronecker delta symbol δ_{IJ}.

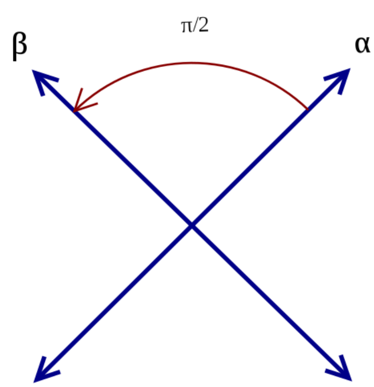
The Lie algebra of SO(4)

The Lie algebra $\mathfrak{so}(4)$ of the Lie group $SO(4)$ is six-dimensional. We will write the general element of $\mathfrak{so}(4)$ as

$$\Omega = \frac{1}{2}\Omega^{IJ} S_{IJ},$$

where S_{IJ} are the generators of the Lie algebra. Note that S_{IJ} is antisymmetric giving the total number of 6 independent generators.

In the classification by Dynkin, it corresponds to the D2 root system:



The root system is thus 2-dimensional and the angle between simple roots is $\pi/2$, which means that

$$\mathfrak{so}(4) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

The two $\mathfrak{su}(2)$ parts of $\mathfrak{so}(4)$ are called the *self-dual* and *anti-self-dual* components. These are given by the projection operators $P^i_{\pm IJ}$:

$$\Omega^i_\pm = P^i_{\pm IJ}\Omega^{IJ}.$$

The explicit expression for the projections is given by:

$$P^i_{\pm jk} = \frac{1}{2}\varepsilon^i_{jk}, \quad P^i_{\pm 0j} = \pm\frac{1}{2}\delta^i_j.$$

The representation theory of $\mathfrak{so}(4)$ follows immediately: irreducible representations are given by pairs of spins:

$$\rho = (j_+, j_-)\,.$$

We will denote irreducibles of $\mathfrak{so}(4)$ by ρ, while irreducibles of $\mathfrak{su}(2)$ are called *spins* and denoted by j.

The two Casimirs of $\mathfrak{so}(4)$ are

$$C = S^{IJ}S_{IJ} = C_+ + C_-,$$
$$\bar{C} = \varepsilon_{IJKL}S^{IJ}S^{KL} = C_+ - C_-,$$

where $C_\pm$ are the $\mathfrak{su}(2)$ Casimirs of the self-dual and anti-self-dual parts.

The simplicity constraint

In 4 spacetime dimensions, General Relativity is described by the Palatini action:

$$S_{GR}[e,A] = \int_M \underline{e}^I \wedge \underline{e}^J \wedge *\underline{\underline{F}}_{IJ} = \frac{1}{2}\varepsilon_{IJKL} \int_M \underline{e}^I \wedge \underline{e}^J \wedge \underline{\underline{F}}^{KL}.$$

This is similar to the BF theory action:

$$S_{BF}[B,A] = \frac{1}{2}\int_M \underline{\underline{B}}_{IJ} \wedge F^{IJ},$$

given that

$$\underline{\underline{B}}_{IJ} = \varepsilon_{IJKL}\underline{e}^K \wedge \underline{e}^L = *\left(\underline{e}_I \wedge \underline{e}_J\right).$$

The precise relation between classical BF theory and classical General Relativity is that GR is equivalent to the BF theory with an additional constraint called the *simplicity constraint*:

> Classical 4D General Relativity can be described by the BF (Plebanski) action
>
> $$S[A,B] = \frac{1}{2}\int_M \underline{\underline{B}}_{IJ} \wedge F^{IJ}$$
>
> with an additional simplicity constraint: B can be written as
>
> $$\underline{\underline{B}}^{IJ} = *\left(\underline{e}^I \wedge \underline{e}^J\right).$$

Can it be that General Relativity, a theory with local degrees of freedom (gravitons), is obtained from the topological theory by further reducing its degrees of freedom? Well, the constraint on B reduces the Lagrange multipliers, not the degrees of freedom. In fact this constraint allows for some degrees of freedom to be dynamical (because their Lagrange multipliers vanish).

It is easy to see that the simplicity constraint implies that

$$\underline{\underline{B}}^{IJ} \wedge *\underline{\underline{B}}_{IJ} = \varepsilon_{IJKL}\underline{\underline{B}}^{IJ} \wedge \underline{\underline{B}}^{KL} = 0.$$

Indeed,

$$\underline{\underline{B}}^{IJ} \wedge *\underline{\underline{B}}_{IJ} = *\left(\underline{e}^I \wedge \underline{e}^J\right) \wedge \left(\underline{e}_I \wedge \underline{e}_J\right) = \varepsilon_{IJKL}\underline{e}^K \wedge \underline{e}^L \wedge \underline{e}^I \wedge \underline{e}^J = 0.$$

Thus, B takes values in a subspace of the Lie algebra $\mathfrak{so}(4)$ defined by the vanishing of the pseudoscalar Casimir

$$\bar{C} = 0.$$

From the results stated in the previous section, we have

$$0 = \bar{C} = C_+ - C_- = j_+\left(j_+ + 1\right) - j_-\left(j_- + 1\right),$$

which gives

$$j_+ = j_-.$$

This can be translated as a restrictions on the representations:

> The simplicity constraint requires B to take values in a subalgebra of $\mathfrak{so}(4)$ for which the pseudo-scalar Casimir vanishes. The representation theory for this subalgebra consists of *simple* representations: the ones for which
>
> $$j_+ = j_-.$$
>
> Simple representations of $\mathfrak{so}(4)$ are labeled with half-integers:
>
> $$\rho = (j,j).$$

Barrett-Crane spinfoams

We wish to enforce the simplicity constraint on the quantum level. This can be achieved through a modification of the TOCY (Turaev-Ooguri-Crane-Yetter) spinfoam model, which can be considered a quantization of the classical topological BF theory.

According to the TOCY spinfoam model, we seek the path integral of the discretized action

$$S[l_f, g_e] = \sum_f \mathrm{tr}\left[g_{e_1} \dots g_{e_k} l_f\right].$$

The partition function of the model is given by the path integral of the action above, while the transition amplitudes are given by the (normalized) partition functions of spinfoams bounded by a chosen spin network state. In the TOCY model, the transition to spinfoam interpretation is given by

$$Z_{\mathrm{TOCY}} = \int Dl \int Dg\, e^{iS[l,g]} = \int Dg \prod_f \delta_{SO(4)}\left(g_{e_1} \dots g_{e_k}\right) =$$

We are trading the continuous path integral over Dl for a discrete sum over irreducible representations with help of the Peter-Weyl theorem.

Now we wish to impose the simplicity constraint dynamically. Thus, we expect the Dl path integral (which is the discrete analogue of DB integral over the $\mathfrak{so}(4)$-valued Lagrange multiplier form B) to only cover those configurations which satisfy the simplicity constraint. We know from the previous section that the fields satisfying the simplicity constraint lie in the subalgebra of $\mathfrak{so}(4)$. Moreover, the functions over this subalgebra can be expanded in the Peter-Weyl series where the irreducible representation runs through the *simple* representations of $\mathfrak{so}(4)$.

We hope to implement the simplicity constraint in the quantum theory by only considering *simple* representations of $\mathfrak{so}(4)$, i.e. those with

$$\rho = (j, j)$$

where j runs through the irreducibles of $\mathfrak{su}(2)$.

The expression for the transition amplitude of the bounding spin network σ in the Barrett-Crane model is thus

$$Z(\sigma) = \sum_{s:\partial s = \sigma} \left(\prod_f \dim\rho_f \prod_v A_v \right),$$

where we sum over spinfoams runs through spinfoams with *simple* representations of $\mathfrak{so}(4)$ associated to edges and the orthonormal basis of intertwiners of these simple representations associated to edges. The vertex amplitude A_v is given by the $\{15\rho\}$ symbol of $\mathfrak{so}(4)$.

Barrett-Crane intertwiner

There is an additional constraint called the cross-simplicity constraint arising from the fact that from

$$\underline{\underline{B}}^{IJ} \wedge * \underline{\underline{B}}_{IJ} = 0$$

it does not necessarily follow that

$$\underline{\underline{B}}^{IJ} = * \left(\underline{e}^I \wedge \underline{e}^J\right)$$

for some frame 1-form field $\underline{e}$. An additional condition is required to hold, which can be expressed as

$$\frac{\varepsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma}}{\varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{MNPQ} B^{MN}_{\alpha\beta} B^{PQ}_{\gamma\delta}} = \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma}.$$

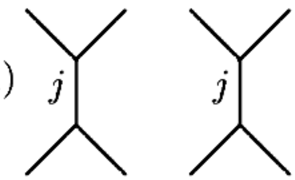
It can be seen (will be explained later) that this expression can be modeled in the quantum theory by fixing the intertwiners associated to spinfoam edges as

$$\iota_{\mathrm{bc}}^{(aa')(bb')(cc')(dd')} = \sum_j \dim j \cdot v^{abf} v^{fcd} v^{a'b'f'} v^{f'c'd'}$$

where the pair (aa') labels two indices in the spin-j irrep of $\mathfrak{su}(2)$, which conspire to give a single index in the $\rho = (j, j)$ irrep of $\mathfrak{so}(4)$. The quantity v^{abc} is either the (unique upon normalization) $\mathfrak{su}(2)$ intertwiner between the three irreps or is equal to zero if a, b and c don't satisfy the quantum version of the triangular inequalities.

The sum goes over the irreps of $\mathfrak{su}(2)$, which are often denoted by a "virtual link" joining the links of the spin network (or a "virtual face" joining faces of the spinfoam):

$$i_{bc} = \sum_j \ (2j+1)$$



Another version of the Barrett-Crane model is then given by the transition amplitude

$$Z(\sigma) = \sum_{s:\partial s=\sigma} \left(\prod_f \mathrm{dim}\rho_f \prod_v A_v \right),$$

where all the intertwiners are fixed to be equal to the Barrett-Crane intertwiner ι_{bc}. The amplitude is still given by the $\{15\rho\}$ symbol of $\mathfrak{so}(4)$, but in this case it only depends on the spins (all the intertwiners are fixed).

Conclusions

- It is not clear which version of the Barrett-Crane model is better suited for describing Euclidean Quantum Gravity in 4D.
- However, both definitely can't be used for describing the physical Lorentzian theory. However, some of the features of Barrett-Crane models are useful when constructing the Lorentzian theory. I hope the 4D Lorentzian Quantum General Relativity to be the subject of my next post.
- The exact form of the Barrett-Crane intertwiner can be constructed by analyzing the properties of a singled out quantum tetrahedron (a spin network node). The spin of the virtual link, as it turns out, corresponds to the dihedral angle of the tetrahedron. I hope to cover this subject later.