3D Loop Quantum Gravity amplitudes

I calculate some transition amplitudes in the Ponzano-Regge spinfoam model, which is a quantization of 3D Euclidean General Relativity. I calculate numerical values for the transition amplitude corresponding to a single spinfoam vertex and investigate its classical limit which turns out to be General Relativity proving a-posteriori that Ponzano-Regge model is indeed a quantization of gravity in 3D. Then results of numerical computations of the renormalized amplitude on the 1-bubble level is given.

The results from this post can be extended to 4D, although the extension is nontrivial. We would have to use a yet-to-be-defined in my blog spinfoam model for 4D Lorentzian Loop Quantum Gravity based on Barrett-Crane spinfoams. Most of the results are similar, but there are exceptions (which I shall emphasize).

Prerequisites

The classical theory of 3D Euclidean General Relativity in Palatini variables is given by the action

$$S[e,\omega] = \frac{1}{2}\,\varepsilon_{abc}\int\limits_M \underline{e}^a \wedge \underline{\underline{F}}^{bc},$$

where $\underline{e}^a = e^a_\mu(x)\,dx^\mu$ is the $\mathbb{R}^3$-valued 1-form called the frame field, $\underline{\omega}^{ab}$ is the $\mathfrak{so}(3)$-valued 1-form called the spin connection and $\underline{\underline{F}}^{ab} = \underline{\underline{d\omega}}^{ab} + \underline{\omega}^{ac} \wedge \underline{\omega}^{cb}$ is the curvature of the spin connection.

The fundamental constant of dimensions of length, which is also the typical scale of quantum gravity, will be denoted

$$\kappa = 8\pi\hbar G$$

in what follows.

Spin networks

Canonical analysis of the Palatini action gives rise to the spin network basis. *Spin networks* are 3-valent graphs with links colored with *spins* — irreducible representations of $\mathfrak{su}(2)$ and nodes colored with *intertwiners* — intertwining operators between the spins of incoming and outgoing links.

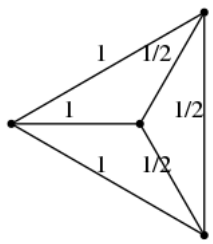
We don't bother specifying intertwiners in the nodes of the graph because of a peculiar property of the recoupling theory of $\mathfrak{su}(2)$: between any three irreps there could be either a single intertwiner or none. In the first case, we don't actually have to specify the intertwiner because it can be derived unambiguously from the three spins. In the second case, there aren't any intertwiners associated to the coloring of links with spins, and we conclude that the spin network doesn't exist.

The important relations, which are required by our demand for the existence of an intertwiner between links with spins $j_{1,2,3}$ are given by:

$$\begin{cases} j_1 \in \{|j_2 - j_3|, |j_2 - j_3| + 1, \ldots, (j_2 + j_3)\}, \\ j_1 + j_2 + j_3 \in \mathbb{R}. \end{cases}$$

These are called the *quantum triangular inequality.*

An example of a spin network:



The links colored with spin-0 irrep are considered nonexistent. Consider a node of the spin network, one of the incident edges of which is nonexistent. Using the quantum triangular inequality it is easy to show that the spins of the remaining two links have to coincide (otherwise, there could be no intertwiner at the node). We can thus erase the node and consider the two links with identical spins to be the same.

The valency of the graph (which is fixed to have 3 incident links to each node, thus valency 3) determines the nature of the dual to the spin network 2-dimensional cellular complex. In fact, valency 3 is precisely what is needed for the dual complex to be *a triangulation*.

Spin network states carry important geometric information: they are eigenstates of 2D *boundary geometry*. To give a precise meaning to this statement, consider a length associated to some piecewise-smooth curve γ embedded in the boundary surface. In the quantum gravity theory with this length we associate a quantum operator acting on the space of states of boundary geometry. Spin networks diagonalize this length operator, the eigenvalues being equal to

$$\text{Length}[\gamma] = \kappa \cdot \sum_a \sqrt{j_a(j_a+1)},$$

where the sum runs over intersections of γ with links of the spin network, j_a labeling the spins of the intersected links. This formula is crucial to the physical interpretation of spin networks: they are eigenstates of the quantum geometry of the boundary surface.

Now consider the boundary triangulation, which is by definition a dual cellular complex to the spin network. To each segment of the triangulation we associate its length given by

$$l_a = \kappa \cdot \sqrt{j_a(j_a+1)} \sim \kappa j_a,$$

where j_a is the spin of the dual link of the spin network. The $\sim$ sign denotes the classical limit of the quantum expression on the l.h.s. By classical limit we traditionally mean large-j limit. It is when quantum numbers are large when the classical behaviour dominates the quantum fluctuations.

This allows for a beautiful physical interpretation of the quantum triangular inequalities. Note that in the classical limit, the first condition gives the classical triangle inequality for the three lengths of an elementary triangle from the triangulation:

$$|l_2 + l_3| \leq l_1 \leq (l_2 + l_3)\,.$$

Thus, quantum triangular inequalities can be interpreted as a quantum version of the well-known triangular inequalities.

Finally, take a second to appreciate the elegancy of the above construction. It truly describes quantized geometry in a fully background-independent way, that is, no reference to any coordinate frame has been used. In the spirit of background independence, quantum geometry arises as a property of quantum states, labeled with the spin network basis. In other words, there is no geometry without a given quantum state. Quantum states themselves determine the geometry instead of living on a fixed background like in QFT in curved spacetime.

The *discreteness* of quantum geometry is also a crucial point. First, let's decide what we mean by discreteness. Even better: let's decide what we don't mean by discreteness: the discreteness introduced by the spin network graph is not what is meant when we say that spin networks describe discrete geometry. What we truly mean when we say that quantized geometry is discrete is — that the eigenvalues of the length operator aren't continuous as one could've expected. Instead there is a minimal possible length introduced by a single intersection with a spin-(1/2) link:

$$l_{\min} = \sqrt{2}\kappa.$$

I claim that the existence of the minimal length in the theory doesn't go against Lorentz invariance. In fact, Lorentz (in our Euclidean case, $\mathfrak{so}(3)$) invariance is gauge and thus crucial to preserve in the quantum theory for reasons of consistency.

One often hears the following argument: suppose we have an extraordinary short stick of length $l = l_{\min}$. What would happen when we Lorentz-contract it? The person asking this question sees two possible answers: it either contracts, or it doesn't. In the former case our original assumption couldn't hold true, since the contracted length is obviously smaller than $l_{\min}$. In the latter case, Lorentz invariance is violated.

This argument is completely wrong though, because it is based on a prejudice of the existence of classical background geometry. In the quantum background-independent theory, on the contrary, the argument fails. The answer to the question (what happens when we Lorentz-contract the stick?) is the following: we get a mixed state with the length of the state being in a superposition of 0 and $l_{\min}$ (and probably other possible eigenvalues). The mean length has indeed contracted and is now slightly less than $l_{\min}$. This doesn't contradict our original assumption, since if we decide to measure the length of the stick, we would get either 0 (the stick disappeared upon measurement), or we would get our $l_{\min}$ back.

The problem of time

General Relativity is a fully constrained theory. It means that its Hamiltonian description consists solely of constraints, the Hamilton's function being zero. Classical Hamiltonian analysis of the Palatini action gives 6 constraints which we can divide in 3 groups:

Constraint group	Symmetry enforced by constraints	Formula for the constraint
Gauss constraints	$\mathfrak{so}(3)$ "Lorentz" gauge invariance	$D_\mu e^\mu_a = 0$
Diffeomorphisms	2-dimensional diffeomorphisms of the boundary	$\frac{1}{2}\varepsilon^{\ c}_{ab} F^{ab}_{\mu\nu} e^\nu_c = 0$
Hamiltonian constraint	the remaining diffeomorphism of 3D theory	$F^{ab}_{\mu\nu} e^\mu_a e^\nu_b = 0$

To build the quantum theory we have to promote constraints to quantum operators acting on the Hilbert space and then solve them by extracting a subspace of *physical states* which solve all the constraints.

The spin network basis solves the first two groups (Gauss constraints and 2D diffeomorphism constraints). The reason for this is that these constraints enforce symmetries which preserve the boundary. Hamiltonian constraint, on the other hand, enforces the remaining diffeomorphisms of the 3D theory which shift the boundary "spatial" surface and thus give rise to time evolution.

In LQG we introduce a notion of the *kinematical Hilbert space* $\mathcal{K}$ (actually it is a rigged Hilbert space, but I omit these technical details here). The states in $\mathcal{K}$ are called *kinematical states*. They contain quantum information about partial observables, that is, background-independent functionals of the boundary geometry. However, they completely fail to capture the quantum dynamics of General Relativity.

We would like to describe the quantum dynamics of the background-independent theory. As a first step, consider quantum dynamics of a field theory on a fixed background.

Intuitively, the dynamics in the quantum theory is given by the Hamiltonian evolution of quantum states. We split the boundary state in the "past" and "future" states:

$$|\Psi\rangle = |\Psi_{\text{past}}\rangle \otimes |\Psi_{\text{future}}\rangle .$$

The dynamics of the quantum theory is captured by the transition amplitude

$$U\left(|\Psi\rangle, t\right) = \langle\Psi_{\text{future}}| \hat{U}(t) |\Psi_{\text{past}}\rangle .$$

If we try to naively employ this technique in the background-independent context, we would run into several interrelated problems:

1. The time t on which the evolution operator depends is given with respect to the background. It is the time which passed relative to the background metric. We don't have t in the background context. But on the other hand, transition amplitudes obviously depend on the time passed!
2. The operator $\hat{U}$ is to be determined from canonical quantization of the Hamiltonian mechanics, but, as we have learned already, the Hamilton's function of General Relativity is identically equal to zero. Instead, the classical dynamics is described by constraints. Naively it appears like there isn't any time evolution at all. Are the states of quantum gravity frozen in time?

As we see, these problems are all connected to the difference of treatments of the concept called "time" in General Relativity (and all background-independent theories) on the one hand, and in Quantum Mechanics on another. The two notions of time are different creating the famous conceptual challenge for any theories which attempt to reconcile the two: *the problem of time*.

The solution to the problem of time lies in the simple yet crucial fact: in General Relativity, as well as in any other background-independent theory, time is just as physical as any other element of our reality. In fact, time is a specific property of the gravitational field! However, there is a coordinate-dependent evolution parameter t which is sometimes also loosely called time. We will call it *coordinate time* or *evolution parameter*, on the other hand we will call the observable function of the gravitational field *physical time*. Hence we make a distinction between the two notions of time.

Which one of the two notions is used in quantum mechanics? Clearly, the quantum mechanical formalism makes use of the coordinate time. Physical time is too complicated to be described by a single real number. This explains why the evolution of states is given solely in terms of constraints: we inspect the evolution law with respect to unphysical evolution

parameter, or coordinate time t. It is not that evolution is frozen, instead we are observing that it is independent of the choice of t. From this point of view it seems inevitable that we would arrive at the problem of time!

How would we describe evolution in the background-independent setting?

Well, we could solve the remaining Hamiltonian constraint of General Relativity and arrive at the (rigged) Hilbert space of *physical states* $\mathcal{H}$. Unlike the ordinary quantum mechanics on the background, these states describe the whole histories of quantum spacetime, not instantaneous information at the given moment of background time.

Let's formulate our solutions to the technical problems formulated above:

1. The time t is unphysical, coordinate time. It can't enter the expression for the transition amplitude simply because the theory is manifestly independent of the choice of t.
2. The operator $\hat{U}$ doesn't depend on unphysical t. The evolution still happens in time, though. But this is physical time, which is a gravitational observable by itself. It means that the quantum information about the passage of physical time is contained in the quantum state, along with all the other properties of the quantum gravitational field. The evolution operator is thus unique, instead of being parametrized by background time t. It describes transitions between kinematical states, with the physical time being encoded in the states themselves.

Spinfoams

How would we find such an evolution operator?

Instead of solving the quantum version of the Hamiltonian constraint (the Wheeler-deWitt equation) we attempt to find a *projection operator* $\hat{P}$ acting on the kinematical space $\mathcal{K}$, such that for any kinematical boundary state its projection satisfies the Wheeler-deWitt equation:

$$\hat{C}\hat{P} = 0,$$

where $\hat{C}$ is the (appropriately defined) quantum operator associated to the Hamiltonian constraint. Formally, the expression for $\hat{P}$ is given by the integral

$$\hat{P} = \int d\tau \, e^{i\hbar^{-1}\tau\hat{C}}.$$

Now comes the trivial yet important observation: our projection operator is nothing else but the evolution operator from the previous section. Indeed, if we want to calculate the amplitude between two kinematical states, we have to project them on the physical subspace and take the physical inner product:

$$(\Phi|\Psi) = \langle\Phi|\,\hat{P}^\dagger\hat{P}\,|\Psi\rangle = \langle\Phi|\,\hat{P}\,|\Psi\rangle\,,$$

where we use $|\Psi) = \hat{P}\,|\Psi\rangle$ to denote physical states, and we have *idempotence* $\hat{P}^\dagger\hat{P} = \hat{P}^2 = \hat{P}$ for any projection operator.

How would we find the matrix elements of $\hat{P}$ between spin networks then?

An interesting covariant approach to the problem emerged in the late 90s called *spinfoam quantization*. In the spirit of path integrals, we want to make sense of the Hawking's "gravitational path integral"

$$\int Dg \, \exp\left(i\hbar^{-1}S_{\text{Einstein-Hilbert}}[g]\right)$$

by discretizing the classical variables on the triangulation and its dual 2-skeleton. This eventually leads to Ponzano-Regge spinfoam m

Spinfoams are higher-dimensional counterparts of spin networks. The standard terminology is to call the elements of the spinfoam cellular complex *vertices*, *edges* and *faces*. Note that each edge is incident to exactly three faces — because of the requirement of being dual to a triangulation. A face, however, can be bounded by any number of edges. In other words, spinfoams are not triangulations, but dual to triangulations.

Spinfoams are colored with irreps and intertwiners of the corresponding gauge algebra (in the case of quantum gravity in 3D, the relevant algebra is $\mathfrak{su}(2)$) as follows. Irreps (spins) are associated to faces of the spinfoam. Intertwining operators are associated to edges, which makes sense because each edge is incident to exactly three faces.

The boundary of the spinfoam is a spin network. Faces give rise to links of the spin network, while edges give rise to spin network nodes. The group-theoretical labels are simply copied to the bounding spin network.

The most important result of the Ponzano-Regge spinfoam model is the expression for the spinfoam amplitude

$$Z(s) = \sum_{\sigma:\,\partial\sigma=s} \left(\prod_f (2j_f + 1) \cdot \prod_v \{6j\} \right),$$

where the sum is taken over spinfoams bounded by a particular spin network s. Here j_f is the spin of an (internal) face f, and $\{6j\}$ is the Wiegner $6j$-symbol for the $\mathfrak{su}(2)$ Lie algebra.

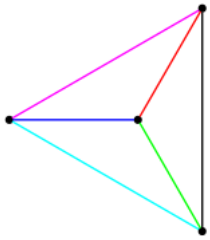
This is our transition amplitude. It describes the amplitude of experiencing a particular spin network. All dynamical predictions of 3D quantum gravity can be obtained solely from this expression.

The interpretation of the sum over spinfoams is two-fold. It can either be interpreted literally as a sum over two-complexes colored with faithful (nontrivial) irreps of $\mathfrak{su}(2)$, or (which is more convenient for our purposes) it can be interpreted as a sum over all possible colorings of a given 2-complex. In the latter case, the colorings with trivial irreps correspond to simpler 2-complexes with some of the faces "switched off", in the same manner as we interpret trivially-colored links of spin networks as "switched off".

First-order computation

I present calculations of amplitudes for a particular class of spin networks in the first order (considering only spinfoams with a single vertex).

A typical spin network of interest is presented on the picture below:



It is based on the full graph with 4 nodes called K_4. The spins of the 6 links are denoted by j, j, j, j, j, j respectively. My choice of labeling links with colors is probably not the best one (especially if the reader uses hardware incapable of displaying colors), but it appears to be the most intuitive.

Each node is given a unique intertwiner provided that quantum triangle inequalities for all four nodes hold. Otherwise, the specified spin network simply doesn't exist in the spin network basis spanning the kinematical space of states $\mathcal{K}$.

Each node can be thought of as a chunk of 2-dimensional boundary. Each link can be thought of as a piece of a mutual boundary of the two incident nodes. Joining chunks in this way gives the topology of a sphere. That is our intuitive treatment of the quantum geometry specified by the spin network K_4: it describes the quantum geometry of a spherical boundary in the 3D differential "spacetime" manifold.

What about the transition amplitude $Z(s)$ associated to the spin network? The spinfoam model suggests a value of

$$\{6j\} = \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}$$

as a first-order approximation. This is given by a single spinfoam which is constructed by putting a spinfoam vertex in the middle of the tetrahedron, joining the nodes of the spin network with the spinfoam vertex with spinfoam edges, and finally, filling the surfaces bounded by spin network links with spinfoam faces. The resulting spinfoam is presented on the picture below (colors changed):

The black lines end on the 4 nodes of the spin network graph K_4.

In what follows we only consider spin networks with equal spins:

$$j = j = j = j = j = j = j.$$

The spin network only exists for integer spins, because otherwise the second quantum triangle inequality isn't satisfied.

The value of the $\{6j\}$ symbol can be calculated in Wolfram Mathematica using the following function:

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SixJSymbol[{j1, j2, j3}, {j4, j5, j6}]
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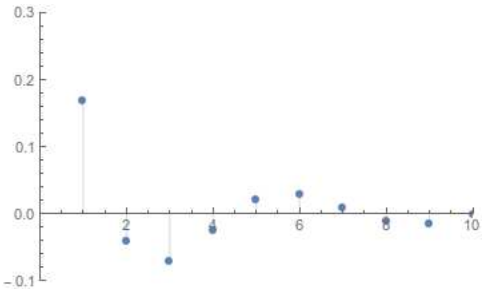
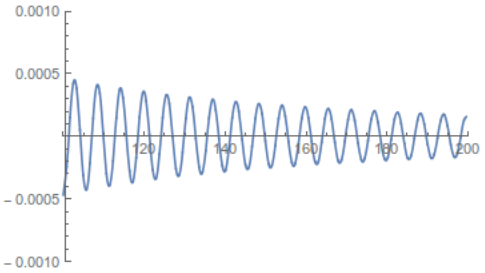
Let's compute $Z(s)$ for spin networks with small spins ($j = 1 \dots 10$). This gives the transition amplitude in the ultra-quantum regime:



Now let's compute the classical limit of $Z(s)$, that is, the value of $Z(s)$ for large spins ($j = 100 \dots 200$):



We observe oscillating behaviour, which is great news and exactly what we hoped for. In the classical limit transition amplitudes are always dominated by the rapidly changing phase (it is called the WKB approximation). The phase of the oscillation is given by a quantity called effective action.

Lets determine the effective action directly from the spinfoam. We are going to make an educated guess, which is: the classical limit of the Ponzano-Regge model is General Relativity in 3D. Discretized action of General Relativity is well-known to be the Regge action

$$S_{\text{Regge}}[l_a, \delta_a] = k \sum_a l_a \delta_a,$$

where the sum is taken over the segments of triangulations (in general number of spacetime dimensions, the sum is taken over $(d-2)$-simplexes or *hinges*), l_a is the length of the a-th segment and δ_a is called the dihedral angle.

We are going to use the following formulas:

$$l_a = \kappa \cdot \sqrt{j_a(j_a+1)} \sim \kappa j_a,$$

$$\delta_a = \arccos\left(\frac{1}{3}\right).$$

The first is the length of the segment of the triangulation, evaluated using the intersections with the spin network (in the case of triangulation segments, we always have a single intersection). The second is the dihedral angle of the regular tetrahedron.

The Regge action thus depends on j as follows:

$$S_{\text{Regge}}(j) \sim 4k\kappa \arccos\left(\frac{1}{3}\right) j = \alpha j.$$

Now we can make a prediction. Suppose that the classical limit of Ponzano-Regge is in fact 3-dimensional General Relativity. Then the classical behaviour of the oscillating phase has to be given by Regge action. Hence we are to expect the oscillating phase to grow linearly with j for large j. This is in correspondence with numerical results! For example, the period of the oscillating curve close to $j = 1000$ and $j = 2000$ is the same (approximately $\Delta j \approx 5.5$).

This result is among the most beautiful insights of Loop Quantum Gravity, along with the quantization of geometric observables. We've seen that the classical limit of the oscillating phase of the quantum transition amplitude is given by the classical discretized action for General Relativity. Thus we have numerically confirmed the hypothesis that the classical limit of 3D Euclidean LQG is General Relativity. The quantum dynamics of 3-dimensional General Relativity is encoded in the group-theoretical construction called the 6j symbol.

Note that we have a real-valued oscillating function $Z(j) \sim \cos(\varphi(j))$ instead of a complex-valued exponent $Z(j) \sim \exp(i\varphi(j))$. It is because we are considering General Relativity in Palatini variables. For any boundary there are two classical configurations instead of one, these are given by different signs of the tetrad. Note that Palatini action changes sign under space-time reflections, giving

$$Z(j) \sim \exp(i\varphi(j)) + \exp(-i\varphi(j)) \sim \cos(\varphi(j)).$$

Bubble computation and renormalization

Now we would like to go beyond first-order. Consider a particular cellular complex given by replacing the single spinfoam vertex from the previous section with a tetrahedron of four spinfoam vertices.

We denote the spins of the 4 internal faces as a, b, c, d (a quite unusual choice of letters, please be aware). The resulting amplitude is given by the topology of the spinfoam:

$$Z(j) = \sum_{a,b,c,d} (2a+1)(2b+1)(2c+1)(2d+1) \cdot$$

$$\cdot \begin{Bmatrix} j & j & j \\ a & b & c \end{Bmatrix} \begin{Bmatrix} j & j & j \\ a & b & d \end{Bmatrix} \begin{Bmatrix} j & j & j \\ a & c & d \end{Bmatrix} \begin{Bmatrix} j & j & j \\ b & c & d \end{Bmatrix}.$$

There is nothing special about writing down the expression above. Its like writing down amplitudes for Feynman diagrams: you simply plug in $\{6j\}$ symbols for spinfoam vertices and keep track of the positions (this can get tricky as the picture is 3-dimensional). I encourage you to try writing down the expression above from scratch.

The sum is over all possible values of spins a, b, c, d and it introduces a divergence. We propose a regularization by a large spin J. We cut off spins of a, b, c, d by requiring them to be less or equal than J.

Numerical computation take a significant amount of time for sensible values of J.

The values of $Z(j)$ computed for small $j = 0, 1, 2$ and $J = 8$ are shown in the table below:

j	$Z(j)$, $J = 8$
0	1785
1	224.61
2	-26.70

The renormalization requires dividing the values by the normalization factor $Z(0)$ representing the amplitude of the null spin network (which is obviously a physical state, as it satisfies the Wheeler-deWitt equation). The following table compares first-order results with renormalized bubble (4th-order) results:

j	First-order $Z(j)$	Renormalized bubble $Z_R(j) = Z(j)/Z(0)$
0	1	1
1	0.17	0.13
2	-0.043	-0.014

Note that first-order terms are already accounted for in the renormalized bubble amplitude, because we are summing over spinfoams some of which have "turned off" faces with spin-0 irreps. These faces simplify the structure of cellular complex naturally giving rise to all kinds of subcomplexes. We are automatically summing over cellular complexes by simply allowing spin-0 irreps.

It turns out that for $J \to \infty$ the renormalized bubble amplitude becomes closer and closer to the first-order amplitude. We can make sense of this by remembering that 3D General Relativity is a topological field theory. It doesn't have propagating degrees of freedom. In spinfoam language it means that refining the triangulation doesn't affect the final result. This is specific to 3D.