ADM decomposition

Consider an arbitrary spacetime metric $g_{\mu\nu}(x)$. In the ADM formalism, it is convenient to switch from the 10 independent components of $g_{\mu\nu}$ to another set of variables:

- $q_{ab}(x)$ is a symmetric rank-2 spatial tensor (with 6 independent components),
- $N_a(x)$ is the shift vector (with 3 independent components),
- $N(x)$ is the lapse function (a single component).

We also introduce the following notation:

- $q^{ab}(x)$ is defined to be the inverse of $q_{ab}(x)$: $q_{ab}q^{bc} = \delta_a^c$,
- $N^a(x)$ is defined to be $N^a = q^{ab}N_b$.

The spacetime metric, by definition of ADM variables, is decomposed as

$$\begin{cases} g_{00} = N_a N^a - N^2 \\ g_{a0} = N_a \\ g_{ab} = q_{ab} \end{cases}$$

The inverse metric is then equal to

$$\begin{cases} g^{00} = -1/N^2 \\ g^{a0} = N^a/N^2 \\ g^{ab} = q^{ab} - N^a N^b/N^2 \end{cases}$$

We now prove that this expression indeed gives the inverse metric, meaning that the equation $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$ holds. The proof consists of four steps:

1. Consider the $\mu = a \neq 0$ and $\sigma = b \neq 0$ case:

$$g_{\mu\nu}g^{\nu\sigma} = g_{a\nu}g^{\nu b} = g_{ac}g^{cb} + g_{a0}g^{0b} = q_{ac}\left(q^{cb} - \frac{N^c N^b}{N^2}\right) + \frac{N_a N^b}{N^2} = q_{ac}q^{cb} = \delta_a^b = \delta_\mu^\sigma.$$

2. Consider the $\mu = a \neq 0$ and $\sigma = 0$ case:

$$g_{\mu\nu}g^{\nu\sigma} = g_{a\nu}g^{\nu 0} = g_{ac}g^{c0} + g_{a0}g^{00} = \frac{q_{ac}N^c}{N^2} - \frac{N_a}{N^2} = 0 = \delta_a^0 = \delta_\mu^\sigma.$$

3. Consider the $\mu = 0$ and $\sigma = b \neq 0$ case:

$$g_{\mu\nu}g^{\nu\sigma} = g_{0\nu}g^{\nu b} = g_{00}g^{0b} + g_{0c}g^{cb} = \left(N_c N^c - N^2\right)\frac{N^b}{N^2} + N_c\left(q^{cb} - \frac{N^c N^b}{N^2}\right) = 0 = \delta_0^b = \delta_\mu^\sigma.$$

4. Finally, consider the $\mu = 0$ and $\sigma = 0$ case:

$$g_{\mu\nu}g^{\nu\sigma} = g_{0\nu}g^{\nu 0} = g_{00}g^{00} + g_{0c}g^{c0} = \left(1 - \frac{N_c N^c}{N^2}\right) + \frac{N_c N^c}{N^2} = 1 = \delta_0^0 = \delta_\mu^\sigma.$$

Thus our proposition $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$ is Q.E.D. and the ADM decomposition indeed gives the correct expression for the inverse metric.

The inverse mapping always exists, meaning that we can compute q_{ab}, N_a and N from $g_{\mu\nu}$. Specifically,

$$\begin{cases} q_{ab} = g_{ab} \\ N_a = g_{a0} \\ N = \sqrt{q^{ab}g_{a0}g_{b0} - N^2} = \left(-g^{00}\right)^{-1/2} = \sqrt{-\frac{\det[g_{\mu\nu}]}{\det[q_{ab}]}} \end{cases}$$