Second quantization in curved spacetime

Consider a free scalar field on the curved background spacetime with metric $g_{\mu\nu}(x)$ of signature $(+,---)$. The action is given by

$$S[\phi] = \frac{1}{2}\int d^4x\sqrt{-g}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi\right),$$

where R is the Ricci scalar of the background metric and ξ is an arbitrary coupling constant. The classical equations of motion are given by

$$\left(\Box + m^2 + \xi R\right)\phi = 0, \qquad \Box = \nabla_\mu\nabla^\mu.$$

The Klein-Gordon bracket

We define the Klein-Gordon bracket on the linear vector space of solutions of the equation of motion:

$$\langle f; g\rangle_\Sigma = i\int\limits_\Sigma d\Sigma\cdot n^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right),$$

where Σ is an arbitrary spacelike hypersurface of dimension $d-1=3$. Note that the bracket is by definition antilinear in its second argument: $\langle f; ag\rangle = a^*\langle f; g\rangle$. Also, $\langle f; g\rangle = \langle g; f\rangle^*$.

Claim: The Klein-Gordon bracket is independent of the choice of Σ as long as both its arguments are solutions of the Klein-Gordon equation.
Proof: Consider a spacetime volume element V enclosed by two spacelike boundaries Σ_1 and Σ_2. The difference between the Klein-Gordon brackets evaluated at the boundaries is equal to

$$\langle f; g\rangle_{\Sigma_1} - \langle f; g\rangle_{\Sigma_2} = i\int\limits_{\Sigma_1} d\Sigma\cdot n^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right) - i\int\limits_{\Sigma_2} d\Sigma\cdot n^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right) =$$

$$= i\int\limits_{\partial V} d\Sigma\cdot n^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right) = i\int\limits_V dV\cdot\nabla^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right) =$$

$$= i\int\limits_V dV\left(\nabla^\mu g^*\cdot\nabla_\mu f - \nabla^\mu f\cdot\partial_\mu g^* + g^*\Box f - f\Box g^*\right) =$$

$$= i\int\limits_V dV\left(g^*\Box f - f\Box g^*\right) = i\int\limits_V dV\cdot g^* f\left(-m^2 - \xi R + m^2 + \xi R\right) = 0.$$

Thus $\langle f; g\rangle$ is independent of the choice of Σ.

Canonical quantization in curved spacetime

Now we specify a foliation of spacetime into spacelike hypersurfaces Σ_t, parameterized by the time coordinate t. By definition, the canonical momentum field is given by

$$\dot\phi = n^\mu\partial_\mu\phi, \qquad \pi = \frac{\partial L}{\partial\dot\phi} = \dot\phi,$$

and the canonical commutation relations read

$$[\phi(\vec{x},t); \pi(\vec{y};t)] = i\delta(\vec{x};\vec{y})$$

with

$$\int\limits_\Sigma dx\cdot\delta(\vec{x};\vec{y}) = 1.$$

The next ingredient is the expression of the field operator and its canonical momentum in terms of the creation and annihilation operators:

$$\begin{cases} \hat{\phi}(x) = \sum_i \left(\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x) \right), \\ \hat{\pi}(x) = \sum_i \left(\hat{a}_i \dot{f}_i(x) + \hat{a}_i^\dagger \dot{f}_i^*(x) \right), \end{cases}$$

where $\{f_i, f_i^*\}$ is a complete orthonormal basis of solutions with positive and negative frequencies defined by

$$\begin{cases} \langle f_i; f_j \rangle = \delta_{ij}, \\ \langle f_i^*; f_j^* \rangle = -\delta_{ij}, \\ \langle f_i; f_j^* \rangle = 0. \end{cases}$$

Claim: $\hat{a}_i = \left\langle \hat{\phi}; f_i \right\rangle$ and $\hat{a}_i^\dagger = -\left\langle \hat{\phi}; f_i^* \right\rangle$.
Proof:

$$\left\langle \hat{\phi}; f_i \right\rangle = \sum_k \left(\hat{a}_k \langle f_k; f_i \rangle + \hat{a}_k^\dagger \langle f_k^*; f_i \rangle \right) = \sum_k (\hat{a}_k \cdot \delta_{ik} + 0) = \hat{a}_i,$$

$$-\left\langle \hat{\phi}; f_i^* \right\rangle = -\sum_k \left(\hat{a}_k \langle f_k; f_i^* \rangle + \hat{a}_k^\dagger \langle f_k^*; f_i^* \rangle \right) = -\sum_k \left(0 - \delta_{ij} \cdot \hat{a}_k^\dagger \right) = \hat{a}_i^\dagger.$$

The operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ indeed have the meaning of creation and annihilation operators, which can be proven by calculating their commutator.

Claim: $\left[\hat{a}_i; \hat{a}_j^\dagger \right] = \delta_{ij}$ and $[\hat{a}_i; \hat{a}_j] = \left[\hat{a}_i^\dagger; \hat{a}_j^\dagger \right] = 0$.
Proof:

$$\left[\hat{a}_i; \hat{a}_j^\dagger \right] = -\left[\left\langle \hat{\phi}; f_i \right\rangle; \left\langle \hat{\phi}; f_j^* \right\rangle \right] = -i^2 \int\limits_\Sigma dx \int\limits_\Sigma dy \cdot \left[f_i^*(\vec{x}) \hat{\pi}(\vec{x}) - \hat{\phi}(\vec{x}) \dot{f}_i^*(\vec{x}); f_j(\vec{y}) \hat{\pi}(\vec{y}) - \hat{\phi}(\vec{y}) \dot{f}_j(\vec{y}) \right] =$$

$$= \int\limits_\Sigma dx \int\limits_\Sigma dy \left(f_i^*(\vec{x}) \cdot \dot{f}_j(\vec{y}) - \dot{f}_i^*(x) \cdot f_j(\vec{y}) \right) \delta(\vec{x}; \vec{y}) = \int\limits_\Sigma dx \cdot n^\mu \left(f_i^* \partial_\mu f_j - f_j \partial_\mu f_i^* \right) = \langle f_i; f_j \rangle = \delta_{ij},$$

and the remaining three commutators can be evaluated in the similar fashion. Here we have used that $\hat{\pi} = \dot{\hat{\phi}} = n^\mu \partial_\mu \hat{\phi}$.

Thus a collection of $\{f_i, f_i^*\}$ with appropriate Klein-Gordon bracket relations is sufficient to manufacture a Bose-second-quantized Hilbert (Fock) space of symmetrized products of creation operators acting on the vacuum $|0\rangle$, which is defined by

$$\forall i: \quad \hat{a}_i |0\rangle = 0.$$

Bogolubov expansions

Consider a curved spacetime with two assimptotical flat boundaries called "the past" Σ_p and "the future" Σ_f. On each boundary there is a preferred choice of the basis of solutions of the Klein-Gordon equation (dictated by the local spatial Fourier transform). We will call these choices $\{f_i; f_i^*\}$ for "the past" and $\{F_i; F_i^*\}$ for "the future". Both of them (by definition) satisfy the usual Klein-Gordon bracket relations.

Since both of them span the linear space of classical solutions, one of them can be expanded in terms of another:

$$\begin{cases} f_i = \sum_k \left(\alpha_{ik} F_k + \beta_{ik} F_k^* \right), \\ f_i^* = \sum_k \left(\alpha_{ik}^* F_k^* + \beta_{ik}^* F_k \right), \end{cases}$$

where α_{ik} and β_{ik} are called the Bogolubov coefficients.

Claim: The physical meaning of the Bogolubov coefficients is given by

$$\alpha_{ik} = \langle f_i; F_k \rangle, \qquad \beta_{ik} = -\langle f_i; F_k^* \rangle.$$

Proof:

$$\langle f_i; F_k \rangle = \sum_p \left(\alpha_{ip} \langle F_p; F_k \rangle + \beta_{ip} \langle F_p^*; F_k \rangle \right) = \sum_p \alpha_{ip} \delta_{pk} = \alpha_{ik},$$

$$-\langle f_i; F_k^* \rangle = -\sum_p \left(\alpha_{ip} \langle F_p; F_k^* \rangle + \beta_{ip} \langle F_p^*; F_k^* \rangle \right) = -\sum_p \beta_{ip} \cdot (-\delta_{pk}) = \beta_{ik}.$$

The Klein-Gordon bracket relations impose the following consistency conditions on the Bogolubov coefficients:

$$\begin{cases} \sum_k \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* \right) = \delta_{ij}, \\ \sum_k \left(\alpha_{ik}\alpha_{jk} - \beta_{ik}\beta_{jk} \right) = 0. \end{cases}$$

Claim: The consistency relations mentioned above hold automatically.
Proof:

$$\delta_{ij} = \langle f_i; f_j \rangle = \sum_{k,p} \left\langle \alpha_{ik} F_k + \beta_{ik} F_k^*; \alpha_{jp} F_p + \beta_{jp} F_p^* \right\rangle =$$

$$= \sum_{k,p} \left(\alpha_{ik}\alpha_{jp}^* \cdot \delta_{kp} - \beta_{ik}\beta_{jp}^* \cdot \delta_{kp} \right) = \sum_k \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* \right),$$

and the computation for $0 = \left\langle f_i; f_j^* \right\rangle$ can be carried out in a similar fashion.

The same logic can be applied to obtain an inverse expansion.

Claim: The inverse expansion is given by

$$\begin{cases} F_k = \sum_i \left(\alpha_{ik}^* f_i - \beta_{ik} f_i^* \right), \\ F_k^* = \sum_i \left(\alpha_{ik} f_i^* - \beta_{ik}^* f_i \right). \end{cases}$$

Proof:

The field operator $\hat{\phi}(x)$ can be expanded in terms of the past and future modes:

$$\hat{\phi}(x) = \sum_k \left(\hat{a}_k f_k(x) + \hat{a}_k^\dagger f_k^*(x) \right) = \sum_k \left(\hat{b}_k F_k(x) + \hat{b}_k^\dagger F_k^*(x) \right).$$

We can relate the $\hat{a}$ and $\hat{b}$ operators by substituting the Bogolubov expansion for $f_k(x)$ and $f_k^*(x)$.

Claim:

$$\begin{cases} \hat{a}_i = \sum_k \left(\alpha_{ik}^* \hat{b}_k - \beta_{ik}^* \hat{b}_k^\dagger \right), \\ \hat{b}_k = \sum_i \left(\alpha_{ik} \hat{a}_i + \beta_{ik}^* \hat{a}_i^\dagger \right). \end{cases}$$

Proof:

$$\hat{a}_i = \left\langle \hat{\phi}; f_i \right\rangle = \sum_k \left(\langle F_k; f_i \rangle \hat{b}_k + \langle F_k^*; f_i \rangle \hat{b}_k^\dagger \right) = \sum_k \left(\alpha_{ik}^* \hat{b}_k - \beta_{ik}^* \hat{b}_k^\dagger \right),$$

and the expression for $\hat{b}_k$ can be carried out in a similar fashion.

Particle creation by the FLRW Universe

Suppose that the Universe contained no particles in "the past". Thus, we are assigning to it a quantum state $|0_p\rangle$ which is a vacuum state for "the past":

$$\forall i: \quad \hat{a}_i |0_p\rangle = 0.$$

We are interested in the quantum expectation of the total number of particles in the mode k present in the Universe in "the future": those particles we interpret as created by the gravitational field:

$$\hat{N}_k = \hat{b}_k^\dagger \hat{b}_k.$$

The quantum expectation is then equal to

$$\langle N_k \rangle = \langle 0_p | \hat{N}_k | 0_p \rangle = \langle 0_p | \hat{b}_k^\dagger \hat{b}_k | 0_p \rangle = \sum_{i,j} \langle 0_p | \left(\alpha_{ik}^* \hat{a}_i^\dagger + \beta_{ik} \hat{a}_i \right) \left(\alpha_{jk} \hat{a}_j + \beta_{jk}^* \hat{a}_j^\dagger \right) | 0_p \rangle =$$

$$= \sum_{i,j} \langle 0_p | \beta_{ik} \hat{a}_i \beta_{jk}^* \hat{a}_j^\dagger | 0_p \rangle = \sum_{i,j} \beta_{ik} \beta_{jk}^* \langle 0_p | \hat{a}_i \hat{a}_j^\dagger | 0_p \rangle = \sum_{i,j} \beta_{ik} \beta_{jk}^* \delta_{ij} = \sum_i |\beta_{ik}|^2 .$$

Now we investigate how the cosmological Friedman-Lemaitre-Robertson-Walker solution of Einstein's equations creates spin-0 particles from the vacuum. The metric is given by

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = dt^2 - a^2(t) \left(dx^2 + dy^2 + dz^2 \right) = a^2(\eta) \left(d\eta^2 - dx^2 - dy^2 - dz^2 \right),$$

where $\eta(t) = t/a(t)$ is called the conformal time.

First, we have to specify a basis of positive norm solutions:

$$f_k(\eta, \vec{x}) = \frac{e^{i\vec{k}\vec{x}}}{a(\eta)(2\pi)^{3/2}} \cdot \chi_k(\eta),$$

where $\chi_k(\eta)$ is a solution of the differential equation

$$\frac{d^2 \chi_k(\eta)}{d\eta^2} + \left(\vec{k}^2 - V(\eta) \right) \chi_k(\eta) = 0$$

with

$$V(\eta) = -a^2(\eta) \cdot \left(m^2 + \left(\xi - \frac{1}{6} \right) R(\eta) \right).$$

Claim: $f_k(\eta, \vec{x})$ indeed give the positive norm solutions of the Klein-Gordon equation $\left(\Box + m^2 - \xi R \right) \phi(x) = 0$.
Proof: OMG this calculation is so complicated I better skip it and hope that this result is indeed correct.

The perturbative calculation for $m = 0$ and around $\xi = 1/6$ gives the following result for the particle-creation Bogolubov coefficient:

$$\beta_{kk'} \approx -\frac{i\delta_{kk'}}{2\omega} \int_{-\infty}^{+\infty} d\eta e^{-2i\omega\eta} V(\eta).$$

For the mean energy density we obtain the following expression:

$$\rho = \frac{1}{(2\pi a)^3 a} \sum_{k,k'} |\beta_{kk'}|^2 \approx -\frac{\left(\xi - \frac{1}{6} \right)^2}{32\pi^2 a^4} \int_{-\infty}^{+\infty} d\eta_1 \int_{-\infty}^{+\infty} d\eta_2 \left(\ln \left[\mu \left| \eta_1 - \eta_2 \right| \right] \cdot \frac{d}{d\eta_1} \left[a^2(\eta_1) R(\eta_1) \right] \cdot \frac{d}{d\eta_2} \left[a^2(\eta_2) R(\eta_2) \right] \right),$$

which is independent of arbitrary μ which is put there for the dimensional considerations. Assuming that $\Delta t \ll H^{-1} = \sqrt{12/R}$, we arrive at an approximate answer:

$$\rho \approx \frac{\left(\xi - \frac{1}{6} \right)^2 H^4}{8\pi^2 a^4} \cdot \ln \left[\frac{1}{H \Delta t} \right],$$

$$N \approx \frac{\left(\xi - \frac{1}{6} \right)^2 H^3}{12\pi a^3}.$$

Thus the FLRW Universe creates massless particles from the vacuum, unless the coupling coefficient is at the value fixed by conformal symmetry ($\xi = 1/6$).